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ESSAYS ON INTRAHOUSEHOLD BARGAINING,
RISK-SHARING, AND THE OPTIMAL BALANCE BETWEEN
PRIVATE INSURANCE AND THE WELFARE STATE

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Abstract

This thesis comprises three essays in the field of applied microeconomic theory. In the first essay we present a generalized Nash model of household decision making that does not restrict *a priori* the household's location on the Pareto frontier and that allows the opportunity cost of household membership to influence the intrahousehold allocation of resources. This approach generalizes both the collective and the symmetric Nash models of household decision making. Formally, we derive the restrictions on household demands implied by the generalized Nash model and we show that the collective model, the symmetric Nash model and the traditional (unitary) model of the household are all special cases of the generalized Nash model.

In the second essay we analyze the optimal risk-sharing contract to emerge between two risk averse individuals under repeated double moral hazard. Several interesting properties of the optimal contract emerge. First, the contract is less sensitive to the performance of any single individual than would have been the case under single moral hazard. Second, a well-known condition describing the optimal level of intertemporal consumption smoothing under repeated single agency is generalized to take account of the double incentive problem. In particular, when *both* individuals face binding incentive constraints then the *expectation* of the ratio of person i 's to person j 's marginal utility in period t is strictly greater than the *known* ratio of person i 's to person j 's marginal utility in period $t - 1$, $i, j = 1, 2, i \neq j$.

In the final essay we examine the optimal balance between the provision of income insurance through family networks and provision through the redistributive tax system. We demonstrate that even when there is full risk-sharing within the family there are nevertheless further welfare gains to be achieved through an appropriate level of government intervention. We also demonstrate that where intra-family moral hazard implies that only partial risk-sharing is achieved within the family, the existence of further welfare gains from government intervention will depend on the effects of such intervention on the intra-family income transfer and on effort incentives.

Chapter 1

Introduction

This thesis comprises three essays, each one addressing a different topic in the field of applied microeconomic theory. The first essay (presented in Chapter 2) looks at the decision making processes within households and attempts to improve upon the way this question has been treated in the conventional economics literature to date. The second essay (presented in Chapter 3) develops a model of risk-sharing between two individuals when there is repeated interaction and double-sided moral hazard. The third and final essay (presented in Chapter 4) examines the question of the optimal balance, from a social welfare point of view, between the provision of income insurance through family networks *versus* provision through a compulsory public insurance scheme (i.e. the redistributive tax system or welfare state).

Although each essay represents an independent and self-contained piece of work and can be read without recourse to any of the other essays, there are nevertheless three separate, unifying themes running throughout the thesis. The first theme is the role and importance of *groups*, rather than *individuals*, in the proper treatment of certain economic questions. Recognition of the economic implications of group interactions is now gaining widespread currency in the economics literature and each of our essays are in this tradition. An example of this is the literature on intrahousehold resource allocation that addresses one of the main problems of the traditional approach towards the household,

that of treating the household as if it were a single individual maximizing a single neo-classical utility function. In terms of our first theme the first essay is concerned with how to characterize the decision making process within the household, taking account of the group nature of such decisions. In this essay we offer a more general characterization of such processes than has so far been provided by either of the two major intrahousehold decision models, namely the collective model and the symmetric Nash model.

There are several examples of economic relationships or contractual arrangements between individuals (or between individuals and organizations such as the firm) that act as a device for dealing with risk. In chapter 3 we discuss some of these arrangements and the way they are characterized in the theoretical literature, while in chapter 4 we discuss the growing evidence to support the existence of risk-sharing within families. In terms of our first theme the second essay examines the question of risk-sharing between two risk averse individuals and describes features of the optimal risk-sharing contract when these individuals interact repeatedly and when there is double moral hazard. Although this essay is not concerned with intra-family risk-sharing *per se*, interestingly the essay began as an attempt to characterize the nature of risk-sharing within families. It soon became apparent however that any such characterization could have applications well beyond the family context and so in this essay we provide a more general treatment of this question.

With regard to the question of designing government policy for dealing with risk, there is increasing recognition of the problems of formulating such policy on the basis of models that ignore the existence of risk-sharing within certain groups, communities and networks in society. Again in terms of the first theme, our final essay discusses and references the literature in this area and examines the question of the optimal design of the welfare state in light of the presence of income-sharing activities within family networks.

The second theme, shared by our second and third essays, is the problem of *risk* and its various private and government responses. In the second essay individuals get together to form a risk-sharing arrangement because it is in their mutual interest to do

so. This is essentially a private response to the problem and occurs without government intervention. The private solution however is imperfect in welfare terms (i.e. perfect risk-sharing is never achieved) because of the presence of moral hazard. Furthermore because of the presence of *double* moral hazard the private solution is worse, in welfare terms, than the solution that arises when there is only single moral hazard (see for instance proposition 10 in chapter 3). We obtain several interesting features of the optimal contract under repeated double moral hazard and show that there are clear differences between the behaviour of this contract and the behaviour of the optimal contract under repeated single moral hazard. This is important because while there are many real world circumstances that can be characterized by single moral hazard (i.e. the classic principal-agent problem) we argue that there are also several circumstances for which double moral hazard provides a more accurate depiction of reality. (As a result our work in this area suggests an interesting avenue for future research, namely an empirical investigation of the extent to which actual risk-sharing arrangements, in double moral hazard contexts, conform to the sort of risk-sharing that is predicted by our model).

In the third essay the existence of private responses to the problem of risk (this time in the context of simple income-sharing arrangements between family members) is taken as given and we then ask how can government intervention be designed so as to raise social welfare above the level of social welfare that would otherwise be achieved in the absence of government intervention. Note that this question is different from the question posed by other authors as to whether family risk-sharing arrangements can be socially harmful when there are market alternatives. For instance Arnott and Stiglitz (1991) show that under certain circumstances this can indeed be the case and so outlawing the existence of family risk-sharing may sometimes be desirable on social welfare grounds. We argue that there are several problems with outlawing family insurance as a policy initiative and so in our third essay we concern ourselves with the question of how policy can be designed to improve welfare *given* the presence of income-sharing within families. Again we obtain results that are interesting, particularly in light of the on-going political debate

about the appropriate role of the state *vis-a-vis* family institutions. We demonstrate that even in those circumstances where there is full risk-sharing within the family, there are nevertheless further welfare gains to be achieved through appropriate government intervention. We also demonstrate that where the presence of intra-family moral hazard implies that only partial risk-sharing is achieved within the family, there *may* be further welfare gains to be achieved via government intervention. However the existence of such gains will depend crucially on the effect of government intervention on the intra-family income transfer and on effort incentives.

The third theme, this time shared by our first and third essays, pertains to certain questions arising out of the *economics of the family*. Understanding the economic behaviour of families is an important research agenda for several reasons. First all individuals, to a greater or lesser extent, exist within some family group. Their economic choices and behaviour are therefore influenced by the interactions that take place inside this group. This may be true even of individuals who live alone if such individuals maintain economic relations with family members residing elsewhere. (Take, for instance, the remittances sent home by economic migrants, the alimony and/or child support payments made between divorced couples, and the housing and job location choices made by professional couples). There is growing evidence, both in industrialized and developing economies, to suggest that many ‘individual’ decisions are taken not in isolation but within the context of a wider family group. Second, if indeed many individual decisions are taken within such a broader context, then this fact should be taken into account when designing public policies that seek to influence individual behaviour. (Note that the idea of a ‘family’ can be interpreted more broadly as any relevant community, network or group that is relevant to the problem being examined. The point is whether, and to what extent, a person’s belonging to such a group affects their economic behaviour).

In terms of the third theme our first essay argues that a richer theory of household decision making would take account of the formation and dissolution of households, and of the effect that a person’s outside opportunities may have on the internal household

decision process. Also our third essay provides an example of the optimal design of government policy, in the context of policies for dealing with risk, taking into account the contractual arrangements that may already exist within families for coping with risk. In the remainder of this chapter we offer a more detailed summary of each of the three essays, highlighting the main contributions made in each case.

1.1 Household Decision Processes and the Intrahousehold Allocation of Resources

In our first essay we explore existing models of the intrahousehold allocation of resources and we offer a more general characterization of the household decision process than has so far been provided in the literature. The symmetric Nash model of the household represents an improvement over traditional approaches in that it addresses the important question of bargaining within a household. It nevertheless remains a highly restrictive model in that given the preferences of individual household members, the symmetric Nash model effectively specifies *a priori* the precise location of the household on the Pareto frontier. In response to this problem the collective model of the household was offered as a generalization of the simple Nash approach. Under the collective model, the only assumption made is that intrahousehold allocations must be Pareto efficient. In other words no *a priori* restrictions are placed on a household's precise location on the efficiency frontier.

In this essay we argue that one of the main weakness of the collective model is that it takes the existence of the household as given and therefore ignores questions to do with the formation and dissolution of households and with the potential for disagreement within any given household. In doing so it also ignores the effect of individuals' outside economic opportunities on a household's internal decision making process and on the precise allocations that are achieved. However this was precisely one of the strengths of the symmetric Nash model. Through specification of a pair of 'disagreement utilities'

(i.e. the utility payoffs received by household members in the event of a disagreement) the symmetric Nash model effectively included a theory of household dissolution, albeit a very simple theory.

Under the symmetric Nash model a person's disagreement utility is taken to be the utility payoff they would receive if they were not a member of the household and if they acted alone. By including these disagreement utilities the symmetric Nash model allows for a person's outside opportunities to influence the outcome reached within the household. As we shall see the particular specification of the disagreement utilities employed in the literature to date embodies quite strong assumptions about the cultural and legal framework governing the dissolution of a household, about the nature of divorce, about the enforcement of laws, and about individual preferences over their marital state. Nevertheless the development of household models that take account of the potential for disagreement helps to focus attention on the need for a proper treatment of the formation and dissolution of households, which can then provide the basis for obtaining a more appropriate specification of the disagreement utilities.

In this essay we argue that by adopting a *generalized* (as against a *symmetric*) Nash model of intrahousehold decision making we are able to preserve the main strengths of both the collective and the symmetric Nash models. Under a generalized Nash model we take account of the fact that a person's 'bargaining power' can vary as the parameters of the model vary, and that the degree of bargaining power may also differ from one household member to the next. This contrasts with the symmetric Nash model which not only fixes the extent of each person's bargaining power but also equalizes bargaining power across household members. Furthermore under a generalized Nash model a pair of disagreement utilities, dependent on the parameters of the model, are specified. This contrasts with the collective model in which questions to do with disagreement and with the role of outside opportunities are not addressed. We demonstrate that such a generalized Nash approach retains the essential characteristics of the collective approach, i.e. it does not restrict *a priori* the household's location on the Pareto frontier. In

this regard it avoids the main criticism of the symmetric Nash model. Furthermore the generalized Nash approach retains the main strength of the simple Nash model. By specifying a pair of Nash disagreement utilities it allows for the *possibility* of disagreement among household members and for the opportunity cost of a person's household membership to influence any cooperative intrahousehold outcome. A generalized Nash approach therefore, by allowing a 'theory of household dissolution' to be incorporated and to influence household decisions, offers a more general characterization of the household decision process than that offered under the collective framework. Also by not restricting the household's location on the Pareto frontier, the generalized Nash model also offers a more general characterization of the household decision process than that offered under the symmetric Nash model. In this essay we demonstrate this claim formally. Specifically we derive the main restrictions on household demands implied by the generalized Nash model and we show that the collective model, the symmetric Nash model, as well as the traditional (unitary) model of the household are all special cases of the generalized Nash model.

1.2 Risk-Sharing under Repeated Double Moral Hazard

In our second essay we extend existing models of repeated moral hazard and analyze the optimal risk-sharing contract to emerge between two risk averse individuals in a situation of repeated double moral hazard. Under double moral hazard *both* individuals have income distributions that are contingent on their own level of effort, however neither individual can observe the level of effort chosen by the other. Such a generalized framework allows us to address questions related to how the optimal long-term contract deals with the trade-off between an efficient level of risk-sharing and the simultaneous provision of *two* sets of incentives. It is the analysis of these trade-offs, in the context of an infinitely repeated agency problem, that represents the key departure of this essay

from the existing literature. Throughout our analysis we adopt recursive methods in order to establish the key qualitative features of the optimal long term contract.

In characterizing the optimal contract under repeated double moral hazard we examine both the contract's within-period characteristics as well as its evolution from one period to the next. It is a feature of the optimal contract that each individual's current consumption as well as their future expected discounted utility is monotonically increasing in their current level of income. Although we do not offer a full characterization of the optimal contract we identify several interesting properties.

First, note that given the expected discounted utility payoff received by the agent in the current period, the optimal contract consists of the principal's effort, the agent's effort, a single schedule of income transfers and a single schedule of future expected discounted utility payoffs, each of the schedules being dependent upon the realised incomes of *both* individuals. It is a feature of the optimal contract that this contract must offer incentives to both individuals *simultaneously*. As a result we show that such a contract is less sensitive to the performance of any single individual than would have been the case if there were only one person with an incentive problem (as in the case of repeated single agency). This result reiterates the predictions of one-period double agency models, however we show that in the infinitely repeated framework, it applies not only to the within-period incentives on offer, but also to the incentives that are offered over time.

Second, a well-known condition describing the optimal level of intertemporal consumption smoothing under repeated single agency is generalized to take account of the double incentive problem. In the literature on repeated single agency it is shown that the *expected* ratio of person i 's to person j 's marginal utility in any period must always equal the *known* ratio of person i 's to person j 's marginal utility in the previous period, where $i, j = 1, 2, i \neq j$, and only j faces a binding incentive constraint. However when the single agency setting is generalized to take account of the double incentive problem, under the optimal contract there is some deviation from the level of intertemporal consumption smoothing that would have been optimal under repeated single agency. We

show that this deviation can be characterized in terms of its benefits and costs. We also show that when *both* individuals face binding incentive constraints then the *expectation* of the ratio of person i 's to person j 's marginal utility in period t must always be strictly greater than the *known* ratio of person i 's to person j 's marginal utility in period $t - 1$, $i, j = 1, 2, i \neq j$.

1.3 Family Insurance and the Welfare State

In our final essay we examine the optimal balance between the provision of insurance through family networks and the provision of insurance through the redistributive tax system or welfare state. Our approach to this question is to take the existence of intra-family insurance arrangements as given and then ask what level of assistance, if any, should additionally be provided through the welfare state. Specifically we address three main questions. First, what is the effect of an expansion of the welfare state on the level of income transferred between family members and on the overall level of insurance available to individuals? Second, what is the mechanism through which these intra-family income transfers are affected? Third, what role exists for public insurance schemes to increase welfare when insurance is also provided within family networks?

We assume that a simple income-sharing arrangement exists within families that consist of two risk averse people. Each family member receives a stochastic income, and to make the analysis tractable we assume that family members interact only once and that any *ex ante* commitments are binding *ex post*. Unemployed family members receive public transfers from the state while employed family members pay taxes to the state.

In carrying out our analysis we consider three different scenarios. First we examine the simplest case of no moral hazard so that each person's income distribution is exogenously given. Here we show that although increases in the size of the public transfer lead to an unambiguous reduction in the level of income transferred between family members, maximizing social welfare requires the provision of full insurance by the state along

with the complete elimination of intra-family insurance. This result is quite intuitive. Since there are no incentive problems, a risk neutral government can implement income transfers across families up to the point where each individual is guaranteed a constant level of consumption in each state of the world. We also show that the extent to which a more generous welfare state succeeds in crowding out the intra-family income transfer will depend upon the proportion of employed and unemployed people in the economy. For instance if the proportion of employed individuals is large, then a given increase in the size of the public transfer paid to each unemployed will require only a small increase in the tax paid by each employed, in order for the government's budget to remain in balance. Therefore for any given increase in the size of the public transfer, the reduction in the net (after-tax) income of employed individuals will be relatively small. However since the family transfer is just half the difference between family members' realized net incomes (i.e. family members offer each other full insurance) then such changes to the tax and public transfer will have a relatively small crowding out effect on the family transfer. The opposite is true if the proportion of employed individuals in the economy is small.

Under the second scenario we introduce moral hazard so that each person's income distribution now depends upon their chosen level of effort. Here we assume that the government cannot observe effort but that family members are able to observe each other. We show that unless an increase in the transfer received by the unemployed requires, for balancing the government's budget, a *reduction* in the level of the tax paid by the employed *by an equal or greater amount* than the increase in the tax, then such an increase in the public transfer must lead to a reduction in the level of the family income transfer. We also show that when the government cannot observe individual actions, maximizing social welfare requires the state to provide less than full insurance to individuals. Finally we also show that under this scenario, regardless of whether or not family transfers are crowded out by increases in the size of the welfare state, there is always a clear role for public insurance schemes to improve welfare beyond the level

that would be achieved if only family insurance were available. Specifically we show that when insurance is available within family networks, as long as effort is observable within the family then a social welfare maximum can never entail zero government intervention.

Under the third and final scenario, we relax the assumption that effort is observable within families and assume that family members are also unable to observe each others actions. We therefore have a problem of moral hazard both between the government and family, as well as within each family. Under this scenario we show that the optimal welfare state must offer less than full insurance to individuals. However this time the case for having any public insurance scheme at all is not so clear cut. Here we identify conditions under which increasing the size of the welfare state, from a position of no government intervention, will be welfare reducing. Under these conditions a social welfare maximum exists *locally* at the point of no government intervention. This means that for sufficiently small levels of government intervention there is no scope for improving welfare via the redistributive tax system, and hence there is a social welfare (local) maximum that entails the provision of no public insurance. An alternative interpretation of this result is that for sufficiently small levels of government intervention it will be preferable for the government to offer no public insurance at all. The occurrence of this outcome depends on the effect of changes in the welfare state on effort and on the family income transfer.

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Chapter 2

Intrahousehold Allocations: Towards a More General Characterization of the Household Decision Process

2.1 Background and Literature Review

Although the behaviour of households with two or more individuals must be the outcome of *group* decisions, most economic models treat the household ‘as if’ it were a *single agent* maximizing a single, neoclassical utility function. This *traditional* approach¹ and its empirical implications have been extensively analyzed and is now well understood (see Ashenfelter and Heckman, 1974; Barten, 1977; Goldberger, 1967; Leuthold, 1968; and Wales and Woodland, 1976).

Nevertheless, despite its familiarity and simplicity², there are some major drawbacks

¹There is a plethora of terms used in the literature to refer to these single agent models of the household. These include the ‘traditional’ model, the ‘classical’ model, the ‘neo-classical’ model, the ‘common preference’ model and the ‘unitary’ model. In each case they refer, essentially, to household models which involve the constrained maximization, by a single individual, of a single utility function.

²Economists have always analyzed *individual* consumer behaviour in this way and so a direct application to the analysis of household decision problems represented a simple and natural extension. Several examples abound in the literature. For instance, Barnum and Squire (1979) and Singh, Squire and Strauss (1986a, 1986b) apply the traditional approach to analyze the behaviour of agricultural house-

to the traditional model. First, under this approach, the household remains a *black box* in that although we are able to characterize the relationship between a household and the wider economy which represents its environment, we are unable to study the internal decision making process and the allocation of resources inside the household. This is primarily because the traditional model does not take account of the group or collective aspect of the household decision process, nor of the fact that household members may have different preferences³. As Manser and Brown (1980) point out, ‘....if the utility functions of the members of the household differ, then an assumption forcing them into this aggregate framework may not be acceptable in all cases.’ Related to this is the question of household formation and dissolution. The traditional model does not offer a framework within which to analyze these questions.

Second, the traditional model does not allow us to consider whether, and to what extent, a person’s economic opportunities outside the household affect the decisions reached inside. In Manser and Brown (1980) and McElroy (1990), the authors refer to a person’s ‘threat point’ as the maximum utility a person would receive if they were no longer a member of the household. The greater a person’s threat point, the more their preferences are reflected in the household’s overall demand functions. The variables that determine this opportunity cost of household membership may include a vector of prices faced by the individual, their wage and nonwage income (the latter might be broadly interpreted to include the individual’s holdings of stocks, pension funds, parental trust funds or even their expected future inheritance), a person’s employment and promotion prospects linked to their education and state of health, and any other variable that might affect

holds involved simultaneously in production and consumption, while McKay and Taffesse (1994) apply this approach to examine the supply response of agricultural households under various market imperfections. Also the standard tools of optimal tax and benefits analysis, welfare analysis, and labour supply analysis are readily applicable. Examples of such applications include Blundell *et al* (1986), Blundell and Walker (1986), King (1983), Pollak and Wales (1981) and Ray (1982).

³Note that strictly speaking, the traditional approach can be used to examine the allocation of resources within the household. This can be done by including, among the arguments of the single neoclassical utility function, the consumption and leisure demands of each individual household member. However it is the fact that the resulting resource allocation within such a framework is not the outcome of *group* decision making that represents the main criticism of the traditional approach.

an individual's economic opportunities outside the household. Of crucial importance too are the laws (and the extent to which they are enforceable) governing the division of financial assets and the custody of children which come into effect upon the dissolution of a household. Any model of household decision making would ideally incorporate such considerations.

Finally, very little evidence has been found to support one of the main predictions of the traditional model, namely that the nonwage incomes of individual family members do not enter the household demand equations separately. Papers by Altonji, Hayashi and Kotlikoff (1989), Cai (1989), Horney and McElroy (1988), Schultz (1990) and Thomas (1990) have all found evidence against income-pooling within the family.

In recent years several alternative lines of research have developed in response to these concerns about the traditional model. One promising approach has been the application of the tools of *cooperative bargaining* to the analysis of household decision problems. In papers employing this approach (see Brown and Manser, 1977, 1978; Manser and Brown, 1978, 1980; McElroy, 1990; and McElroy and Horney, 1981), overwhelming attention has been given to applying the *symmetric* Nash bargaining solution to the analysis of household decision problems. In these papers the authors examine the resulting Nash system of household demand equations and their comparative static properties, highlighting the main differences between these and the traditional demands and comparative statics. In Manser and Brown (1980), the dictatorial and the Kalai-Smorodinsky bargaining solutions are briefly considered while in Creightney (1997), an egalitarian solution is also examined.

The cooperative bargaining approach enables us to overcome many of the weaknesses of the traditional model. Under cooperative bargaining household members are allowed to have different preferences. Also, a 'disagreement utility' or 'threat point' can be specified for each household member so that the utility payoff to any member under the cooperative bargaining outcome is restricted to being at least as large as a person's disagreement utility. As we argue later, these disagreement utilities need not be arbitrarily chosen

utility payoffs but may be the payoffs predicted by some suitable theory of household dissolution. Finally, the cooperative bargaining approach leads to an alternative to the income-pooling hypothesis.

Under a separate line of inquiry and in response to both the traditional and the cooperative bargaining approaches, Chiappori (1988, 1992) (hereafter referred to as CH) proposed a *collective* approach to analyzing household decision problems. Under the collective approach household members have their own individual preference orderings and the decision process is assumed to be cooperative in that all outcomes are Pareto efficient (CH refers to this as the *efficiency approach*).⁴ However no additional restrictions are imposed *a priori* as to which point on the efficiency frontier the household will ultimately choose. CH also presents an equivalent formulation of the collective decision process as the constrained maximization of a weighted sum of individual utilities, where the weights are taken to vary continuously with the exogenous parameters of the model. We will refer to this formulation of the collective hypothesis as the *(weighted) sum of utilities approach*. Under this sum of utilities interpretation, by allowing each person's utility weight in the collective decision process to vary continuously with the exogenous parameters, all points on the Pareto frontier can be traced out. In this way no particular point is selected *a priori*. Under the collective line of inquiry CH characterizes the set of efficient intrahousehold allocations and develops a set of empirical restrictions that must be satisfied by any system of household demands if the efficiency hypothesis is true.

At this stage we raise two main points about the collective approach and its relationship to other cooperative approaches that employ a specific bargaining solution. First,

⁴Chiappori offers an alternative interpretation of the *efficiency approach* under which the household decision process is essentially a two-stage budgeting one (the *sharing rule approach*). In the first stage, household members allocate the total household nonlabour income among themselves according to a pre-determined sharing rule. In the second stage, once nonlabour income has been allocated, each household member then chooses their own consumption and labour supply through constrained maximization of their own individual utility subject to their individual budget constraint. Under this alternative interpretation, the way in which the sharing rule is determined remains outside the analysis. Chiappori argues that both the efficiency and the sharing rule approaches are equivalent in that household decisions are efficient *if and only if* a sharing rule exists.

as CH points out, the collective approach uses knowledge of individual utilities and the household budget constraint to generate only a continuum of Pareto efficient outcomes. Since no particular outcome is selected, it therefore does not allow the derivation of a system of household demand equations. This is in contrast, say, to the symmetric Nash model which implicitly assumes that household members have fixed and equal weights in the household decision process. Under the symmetric Nash model therefore, a specific point is selected on the Pareto frontier, namely the point that distributes the gains to cooperation equally between the household members. McElroy and Horney (1981) and McElroy (1990) (hereafter referred to as MHM) accordingly derive the corresponding system of symmetric Nash household demands, along with its comparative static properties, in which individuals' utility weights are fixed with respect to changes in the exogenous parameters.

In this chapter we argue, however, that under the sum of utilities interpretation of the collective model it is possible to derive a system of collective household demands in which the individual utility weights, as functions of the exogenous parameters, are included among the arguments of the demand functions. This follows since each pair of utility weights (in a 2– person household) corresponds to a distinct utility allocation (or a distinct consumption allocation if the analysis is carried out in consumption space). In deriving such a system of *conditional* demands for the collective model we are also able to derive the model's comparative static properties, taking into account both the direct effect of parameter changes on household demands as well as the indirect effect that occurs through changes in the individual utility weights.

Second, CH offers a set of restrictions (referred to as restrictions on collectively rational egoistic agents, or CREA) that should characterize any efficient intrahousehold allocation. Since the solution to a symmetric Nash bargain is always efficient, we expect that such a solution will also satisfy CREA. CH also points out that the collective approach of generating only a continuum of efficient outcomes should encompass the symmetric Nash-bargained approach of selecting a particular efficient outcome, and so

the symmetric Nash framework must be a special case of the collective framework. CH therefore conjectures that the symmetric Nash should impose *additional* structure on the system of household demands, beyond that represented by CREA. However no formal proof of this has yet been offered.

In this chapter we offer a more general characterization of the household decision process than has so far been offered under either the collective or the symmetric Nash approaches. We take the symmetric Nash model of MHM and develop a generalized version that explicitly takes account of differences between and variations in the ‘bargaining strength’ of each family member. We demonstrate that such an approach retains the essential characteristics of the collective approach, namely that it does not restrict *a priori* the household’s location on the Pareto frontier. In so doing it avoids one of the main criticisms of the symmetric Nash model. This approach also allows us to derive a system of household demands *conditional upon location* (as is also the case under the collective model). However our generalized Nash approach has the additional advantage of retaining one of the main strengths of the simple symmetric Nash model. Specifically, through specification of a pair of Nash disagreement utilities, the generalized Nash model allows for the *possibility* of disagreement among household members and for the opportunity cost of a person’s household membership to influence any cooperative outcome. A generalized Nash approach therefore, by allowing a ‘theory of household dissolution’ to be incorporated into the intrahousehold resource allocation model and to have some influence over household decisions, as well as by avoiding the *a priori* restriction of the household’s location on the Pareto frontier, offers an even more general characterization of the household decision process than that offered under either the symmetric Nash or the collective frameworks.

This chapter is structured as follows. In section 2.2 we present a simple household bargaining problem with 2 household members, m and f . In section 2.3 we argue that a generalized Nash model (without the specification of a pair of disagreement utilities) is equivalent to the collective model in so far as both models use knowledge of the

individual utilities and of the household budget constraint to generate a continuum of efficient utility allocations, without restricting the household *a priori* to any particular point on the Pareto frontier. In developing this argument we first discuss the set of utility allocations from which the household chooses under the collective and the generalized Nash frameworks. Under the collective model presented in CH, the household chooses a utility allocation from among the entire set of feasible utility allocations. The minimum utility payoffs are therefore just the minimum feasible utility payoffs. However under the symmetric Nash model of MHM (and the generalized version presented later on), a pair of disagreement utilities, interior to the set of feasible utilities, are specified for each household member. These disagreement utilities effectively form the minimum utility payoffs that are acceptable to the household members under any cooperative agreement. Under the generalized Nash model therefore, the household is restricted to choosing from a subset of the set of feasible utilities. We argue that if both the generalized Nash and the collective models specify the same minimum acceptable utilities for each household member, then in both models the household will choose from the same (sub)set of feasible utilities. More specifically, in both models the household will select an allocation from among the set of efficient allocations that lie upon exactly the same portion of the Pareto frontier.

Formally, we state our equivalence result as follows: given individual preferences, a particular representation of those preferences and a set of exogenous parameters, then subject to certain restrictions on the disagreement utilities of the generalized Nash model, the collective and the generalized Nash *classes* of household preference orderings will rationalize the same *set* of household utility allocations. We then present a formal analysis of this point, in utility space. We specify the same, fixed, minimum utility payoff for each household member, and then following Pollak (1977), we formally define what is meant by the rationalization of an allocation (set of allocations) by a preference ordering (class of preference orderings). We then show that both the generalized Nash and the collective classes of household preference orderings must rationalize the same set of efficient utility

allocations, when both models specify the same minimum utility payoffs. Specifically, we show that for every possible efficient utility allocation, there is a *distinct* relative utility weight for which the generalized Nash model will yield this given utility allocation as a solution. Likewise we also show that, for every possible efficient utility allocation, there is a *distinct* relative utility weight for which the collective model will yield this given utility allocation as a solution.⁵

Let μ represent the weight attached to person f 's utility relative to that of person m 's utility in the collective household objective, and let the mapping $\mu(\alpha)$ represent the relationship between μ and a vector, α , containing the exogenous parameters of the household decision problem. Likewise let γ represent the weight attached to person f 's utility relative to that of person m 's utility in the generalized Nash household objective, and let the mapping $\gamma(\alpha)$ represent the relationship between γ and the parameter vector, α . An important corollary of our result is that, for any mapping $\mu(\alpha)$, we can find a mapping $\gamma(\alpha) = \gamma(\mu(\alpha), \alpha)$ such that, for any given parameter vector α , both the collective and the generalized Nash models choose precisely the same utility allocation.

We make two points with regard to this corollary. First, our result on the equivalence of the generalized Nash and the collective frameworks does not hinge on the relative utility weight $\gamma(\mu(\alpha), \alpha)$ being used in the generalized Nash model whenever the relative weight $\mu(\alpha)$ is used in the collective model. By arguing that these two approaches are equivalent we are merely saying that both approaches generate a continuum of efficient household utility allocations and that neither approach restricts the household *a priori* to a particular location on the Pareto frontier. Second, if the generalized Nash utility weights are nevertheless chosen according to $\gamma(\mu(\alpha), \alpha)$, then both models also yield the same particular utility allocation, for any parameter vector α .

In section 2.4 we argue that a generalized Nash approach, when it allows for the *possibility* of disagreement among household members and for the opportunity cost of

⁵Throughout this chapter we work with 2-person household models. 'Relative utility weight', therefore, refers to the utility weight of one family member, relative to that of the other, in the overall household objective.

each person's household membership to influence the cooperative outcome, in fact offers a more general characterization of the household decision process than that offered under the collective model of CH. We approach this formally by examining the properties exhibited by both the collective and the generalized Nash demand systems, and by showing that the collective model of CH in fact entails more structure on household demands than the generalized Nash model presented below.

In developing the analysis we follow the approach of MHM and Manser and Brown (1980) and allow each person's disagreement utility, in the generalized Nash problem, to be given by the maximized value of their individual utility in a private constrained maximization problem. Each disagreement utility therefore depends upon the parameters of the corresponding private optimization problem. Also we follow CH and allow the household, in the collective model, to choose from among the entire set of feasible utilities.

We derive the fundamental matrix equations of comparative statics for both the collective and the generalized Nash models and interpret them, as in MHM, as a complete statement of the empirical content of the respective models. We then derive the main properties of the collective and the generalized Nash demand systems. In particular we obtain Slutsky equations for both models and show that these equations are identical except for the presence of an extra term in the generalized Nash Slutsky equation. This extra term represents the effect of an increase in the price of good k , compensated by an increase in income, on the generalized Nash disagreement utilities, and the consequent effect this has on the intra-household demand for good j , $\forall j, k$. We also show that restricting the generalized Nash model so that its disagreement utilities are constant with respect to any change of parameter implies that the generalized Nash comparative statics collapse to the collective comparative statics and both models become empirically indistinguishable.

In section 2.5 we recap some of the results obtained in MHM who analyzed the traditional and the symmetric Nash models and demonstrated that if the symmetric Nash fundamental equations were restricted so that its disagreement utilities no longer

varied with the exogenous parameters, then the comparative statics of the symmetric Nash demand system would collapse to those of the traditional demand system. The symmetric Nash demand system was therefore shown to generalize the traditional demand system so as to take account of the disagreement utilities and their possible dependence on the exogenous parameters. We combine the results of MHM with our own results to establish the following relationship between all four models: the comparative statics of the traditional model are nested within those of the symmetric Nash model which in turn are nested within those of the generalized Nash model. Also the comparative statics of the traditional model are nested within those of the collective model which in turn are nested within those of the generalized Nash. Section 2.6 concludes this chapter with a discussion of areas for further work.

2.2 A Household Bargaining Problem

We adopt the simplest formulation of a household bargaining problem⁶ and assume that the household consists of 2 members indexed $h = m, f$.⁷ Let x^1 be a good consumed by m , x^2 a good consumed by f , x^3 the leisure time of m and x^4 the leisure time of f . Let x^0 represent a private consumption good which is shared whenever m and f form a household. We take x^0 to represent a pure public good so that consumption of x^0 by one individual does not diminish the amount available for consumption by the other. Then x is the (5×1) vector of household consumption goods and leisure $(x^0, x^1, x^2, x^3, x^4)'$, with p the corresponding (5×1) vector of prices $(p^0, p^1, p^2, p^3, p^4)'$. We refer to x^f as the (3×1) consumption vector of f , $(x^0, x^2, x^4)'$, and x^m as the (3×1) consumption vector of m , $(x^0, x^1, x^3)'$. Finally, we index individual consumption goods and leisure by $i, j, k = 0, 1, \dots, 4$.

⁶Our formulation is essentially the same as that adopted by MHM, CH and by Manser and Brown (1980), although it should be noted that in CH, no shared goods are included within the household members' consumption baskets.

⁷A more general analysis with $n (> 2)$ household members can be developed, e.g. Creightney (1997).

Assumption 1. $x^i \geq 0, \forall i$.

Family member h has nonwage income y^h ($h = m, f$) and a time endowment given by T . Then y is the (2×1) vector of nonwage incomes $(y^m, y^f)'$. Also we define $\alpha \equiv (p', y')$ as a vector containing all the parameters of the household's decision problem. The household's full income constraint is given by

$$p'x \leq (p^3 + p^4)T + y^m + y^f,$$

or equivalently

$$p'q \leq y^m + y^f,$$

where q is the (5×1) vector of excess demands

$$(x^0, x^1, x^2, x^3 - T, x^4 - T)' = (q^0, q^1, q^2, q^3, q^4)'.$$

Individual preferences are represented by the egoistic utility functions $U^m = U^m(x^m) = U^m(x^0, x^1, x^3)$ and $U^f = U^f(x^f) = U^f(x^0, x^2, x^4)$, for m and f respectively. Also U_z^h and $U_{z_1 z_2}^h$ ($h = m, f$) represent, respectively, the first and second partial derivatives of U^h with respect to members of the vector x .

Assumption 2. U^h ($h = m, f$) is twice continuously differentiable; also $U_z^f > 0 \forall z \in x^f$; $U_w^m > 0 \forall w \in x^m$; $U_{z_1 z_2}^f < 0, \forall z_1, z_2 \in x^f$; and $U_{w_1 w_2}^m < 0, \forall w_1, w_2 \in x^m$.

Given the household economy just described, the set of feasible consumption allocations, X , is given by

$$X \equiv \{x \in R_+^5 : y^m + y^f - p'q \geq 0\},$$

while the set of feasible utility allocations, U , is given by

$$U = \{(U^m, U^f) : x \in X\}.$$

The following lemma presents some standard results on cooperative bargaining.

Lemma 1. The set of feasible utility allocations, U , is closed, bounded and strictly convex.

Proof. See Appendix. ■

The next two results demonstrate the role of the shared good, x^0 , in defining a non-degenerate household bargaining problem. We first define the function $V^h(\cdot)$ as person h 's indirect utility function ($h = m, f$). The arguments of this function are just the parameters of person h 's individual constrained maximization problem. We take V^h to represent the value of h 's indirect utility function for any given value of the parameters. Let x_m^0 and x_f^0 represent the quantities of the shared good, x^0 , consumed by m and f respectively when m and f act privately. Then

$$\begin{aligned} V^m &= V^m(p^0, p^1, p^3, y^m) \\ &= \max_{x^i, i=0,1,3} \{U^m(x_m^0, x^1, x^3) / p^0 x_m^0 + p^1 x^1 + p^3 x^3 \leq y^m + p^3 T\} \end{aligned} \quad (2.1)$$

and

$$\begin{aligned} V^f &= V^f(p^0, p^2, p^4, y^f) \\ &= \max_{x^i, i=0,2,4} \{U^f(x_f^0, x^2, x^4) / p^0 x_f^0 + p^2 x^2 + p^4 x^4 \leq y^f + p^4 T\}. \end{aligned} \quad (2.2)$$

In the absence of a shared good the corresponding indirect utilities are given by

$$\begin{aligned} V_0^m &= V_0^m(p^1, p^3, y^m) \\ &= \max_{x^i, i=0,1,3} \{U^m(x_m^0, x^1, x^3) / p^0 x_m^0 + p^1 x^1 + p^3 x^3 \leq y^m + p^3 T \text{ and } x_m^0 = 0\} \end{aligned} \quad (2.3)$$

and

$$V_0^f = V_0^f(p^2, p^4, y^f)$$

$$= \max_{x^i, i=0,2,4} \{U^f(x_f^0, x^2, x^4) / p^0 x_f^0 + p^2 x^2 + p^4 x^4 \leq y^f + p^4 T \text{ and } x_f^0 = 0\}. \quad (2.4)$$

In general the indirect utilities (or disagreement utilities), V^m and V^f , where $(V^m, V^f) \in \mathbf{U}$, are the utilities received by m and f respectively in the event of a disagreement within the joint household. These utilities can be interpreted as the respective individual's privately optimal utility payoff or as their opportunity cost of household membership. In other words V^m and V^f represent the utility received by m and f , respectively, in their next best alternative.⁸ Suppose, in the joint household, individuals are able to reject any household allocation that leaves them less well off than they would otherwise be by acting privately. If this were indeed the case then any household bargain must offer individuals at least their privately optimal utility payoffs. Our next result demonstrates that if there are no shared goods then there can be no 'utility gains' to be had from membership of a joint household over and above the privately optimal utility payoffs just specified.⁹

Lemma 2. Suppose there are no shared goods. Let $(0, x^{1*}, x^{3*})$ and $(0, x^{2*}, x^{4*})$ represent the consumption bundles that are privately optimal for m and f respectively, i.e. $(0, x^{1*}, x^{3*})$ is the solution to (2.3) while $(0, x^{2*}, x^{4*})$ is the solution to (2.4). Let V_0^m and V_0^f represent the corresponding utility payoffs. Then there exists no $(U^m, U^f) \in \mathbf{U}$ such that $U^m > V_0^m$ and $U^f > V_0^f$.

Proof. See Appendix. ■

Note that we must have $V^m \geq V_0^m$ and $V^f \geq V_0^f$, since relaxation of the constraint $x_h^0 = 0$ cannot make person h worse off and may well make him/her better off ($h = m, f$). The next result demonstrates that the presence of a shared good means that there are

⁸Note however that the particular disagreement utilities we have specified in (2.1) - (2.4) are by no means the only candidates for such an interpretation. A different 'model' of household dissolution may well yield different utility levels $(v, w) \in \mathbf{U}$ for individuals when they are no longer members of the joint household. We elaborate on this point later.

⁹This simple formulation of course ignores any non-pecuniary benefits to being a member of a joint household. Implicit in our result is the assumption that individual utilities are independent of the 'marital state'.

utility gains to be derived, from joint household membership, over and above what can be achieved privately.

Lemma 3. Let V^m, V^f, V_0^m and V_0^f be as defined in (2.1) - (2.4). If there is a shared good and there are interior solutions for this shared good when individuals act privately, then there exists $(U^m, U^f) \in \mathbf{U}$ such that $U^m > V^m \geq V_0^m$ and $U^f > V^f \geq V_0^f$.

Proof. See Appendix. ■

Lemmas 2 and 3 are important for demonstrating the importance of including a shared good in our analysis of the household resource allocation problem. If household members are able to reject a joint household allocation in favour of some privately optimal utility payoff, such as the payoffs V^m and V^f , then the existence of a shared good will be both necessary and sufficient to ensure that an interesting bargaining problem exists, i.e. that the solution to the household bargaining problem is not degenerate. We have chosen to illustrate the point with reference to the particular feasible payoffs (V^m, V^f) , defined by (2.1) and (2.2). Note however that all we really require for a non-degenerate bargaining problem is that there exists $(U^m, U^f) \in \mathbf{U}$ such that $U^m > v$ and $U^f > w$, for *some* $(v, w) \in \mathbf{U}$ that represents the household members' 'disagreement utilities', however these disagreement utilities may be defined. If there were no feasible utility allocations that made at least one person better off by joining the household than by acting alone (and no one worse off), then there would be no incentive for household members to cooperate.

2.3 Equivalence of the Generalized Nash and the Collective Approaches to Household Decision Making

In this section we provide support for our claim, made in section 2.1, that given individual preferences, a particular representation of those preferences and a set of exogenous parameters, then subject to certain restrictions on the disagreement utilities of the generalized

Nash model, the collective and the generalized Nash *classes* of household preference orderings will rationalize the same *set* of household utility allocations. In Section 2.3.1, we discuss the household's choice set under the collective and under the generalized Nash frameworks. We argue that if both models specify the same minimum acceptable utilities for each household member, then in both cases the household will choose from the same (sub)set of feasible utilities. More specifically, in both cases the household will select an allocation from among the set of efficient allocations that lie upon exactly the same portion of the Pareto frontier.

In Section 2.3.2 we formalize our analysis and present a formal definition of the rationalization of an allocation (set of allocations) by a preference ordering (class of preference orderings). Under this definition we show that both the generalized Nash and the collective classes of household preference orderings must rationalize the same set of efficient utility allocations, if both models specify the same minimum utilities. We then present a corollary of this result which says that for any relationship between the relative utility weight of the collective model and the exogenous parameters, we can find a relationship between the relative utility weight of the generalized Nash model and the exogenous parameters such that both models always select the same utility allocation for any given parameter vector.

2.3.1 The Choice Set of Feasible and Efficient Utility Allocations

Once individual preferences and the parameters of the problem have been specified, it is obvious that the set of *feasible* utility allocations, U , and the corresponding set of *efficient* utility allocations must be the same under the generalized Nash model as they are under the collective model. Nevertheless, there may be good reasons for excluding some of the allocations in U from consideration when solving the joint household's decision problem. If a joint household allocation leads to an individual receiving a utility payoff that is lower than some privately optimal payoff, then it may be reasonable to expect this allocation to

be rejected by the individual and therefore not to be chosen by the joint household, even if it is feasible. Furthermore we could model a pair of minimum payoffs in such a way as to capture the opportunity cost of household membership, and these payoffs could be the outcome of an appropriate theory of household dissolution. Such an approach would allow us to take account of possible noncooperative behaviour and to examine the way in which this influences the outcome within the cooperative setting.

Under the symmetric Nash model of MHM, a pair of disagreement utilities, interior to U , are specified for each household member. These disagreement utilities are taken to be the utility payoffs that a household member would receive if s/he acted privately, and serve to restrict the set of utility allocations under consideration to a subset of U .

In contrast, under the collective model of CH, no such disagreement utilities are specified and therefore any allocation in U can be considered by the household members when choosing a solution to the household bargaining problem. However if, in the collective model, we specified a payoff for each household member that was individually rational and which then formed that member's minimum acceptable payoff in the joint household agreement, and if these payoffs were taken to be the same as the disagreement utilities in the Nash-bargaining model, then the resulting *restricted* set of feasible utility allocations would be identical in both models. Also, the set of efficient utility allocations corresponding to this restricted set would be the same in both cases.

The Collective Model - Efficiency Interpretation

Under the collective model specified by CH, household behaviour is a solution to¹⁰

$$\max_{x^i, \forall i} \{ U^m(x^m) / U^f(x^f) \geq u^f, y^m + y^f - p'q \geq 0 \}, \quad (C1)$$

for some feasible utility, u^f , where u^f is taken to be a function of the parameters of the model. Under this framework, the household Pareto frontier is defined, for any parameter

¹⁰Here we have modified the model presented in CH to include the shared good, x^0 , among the basket of goods consumed by household members.

vector α , by

$$F(u^f, \alpha) \equiv \max_{x^i, \forall i} \{U^m(x^m)/U^f(x^f) \geq u^f, y^m + y^f - p'q \geq 0\}. \quad (2.5)$$

Given α , the set of efficient household allocations obtains as u^f varies within its domain. Also, as α varies, the shape of the Pareto frontier also varies. Throughout we will use the notation $F'(u^f, \alpha)$ and $F''(u^f, \alpha)$ to refer, respectively, to the first and second partial derivatives of F with respect to u^f , for a given α , i.e. $F'(u^f, \alpha) = F_{u^f}(u^f, \alpha)$ and $F''(u^f, \alpha) = F_{u^f u^f}(u^f, \alpha)$. Also we let $F_w(u^f, \alpha)$ represent the derivative of F with respect to any element w in the parameter vector α .

CH does not specify the domain of u^f but it is clear that under the joint household full income constraint, the maximum feasible utility for f , u_{\max}^f , must be the utility that would accrue to f if m is constrained to zero consumption of x^1 and x^3 , and f maximizes her individual utility subject to the joint budget constraint.¹¹ Then

$$u_{\max}^f \equiv \max_{x^i, i=0,2,4} \{U^f(x^f)/p^0 x^0 + p^2 x^2 + p^4 x^4 \leq y^m + y^f + (p^3 + p^4)T\}. \quad (2.6)$$

If $U^f = u_{\max}^f$ then since, in the solution to (2.6), the budget constraint is binding, we must have $U^m = u_{\min}^m \equiv U^m(x_f^{0*}, 0, 0)$, where x_f^{0*} is the level of x^0 chosen under (2.6). Alternatively, $u_{\min}^m = F(u_{\max}^f, \alpha)$.

By a symmetric argument, the maximum feasible utility for m must be

$$u_{\max}^m \equiv \max_{x^i, i=0,1,3} \{U^m(x^m)/p^0 x^0 + p^1 x^1 + p^3 x^3 \leq y^m + y^f + (p^3 + p^4)T\}, \quad (2.7)$$

while the corresponding minimum feasible utility for f must be $u_{\min}^f \equiv U^f(x_m^{0*}, 0, 0)$, where this time x_m^{0*} is the level of x^0 chosen under (2.7). Alternatively $u_{\min}^f = F^{-1}(u_{\max}^m, \alpha)$.¹²

¹¹Note that although there are other consumption allocations for f that would yield the same utility as u_{\max}^f , these would not be affordable given the strict concavity of U^f .

¹²Given any parameter vector α , the function describing the Pareto frontier, $F(u^f, \alpha)$, is invertible since, as we will show later, it is monotonically decreasing in u^f , i.e. $F'(u^f, \alpha) = F_{u^f}(u^f, \alpha) < 0, \forall u^f$.

Furthermore, since $U^f(ax^f + (1-a)\hat{x}^f)$ is continuous in a , $\forall a \in [0, 1]$ and for any two feasible consumption allocations x^f and \hat{x}^f , then $U^f(ax^f + (1-a)\hat{x}^f)$ varies continuously from $U^f(\hat{x}^f)$ to $U^f(x^f)$ as a varies from 0 to 1.

Therefore the set of feasible utilities u^f must be the compact interval $[u_{\min}^f, u_{\max}^f]$, while the set of feasible utilities u^m must be the compact interval $[u_{\min}^m, u_{\max}^m]$. We can therefore equivalently define the set of feasible utility allocations, U , as

$$U = \left\{ (u^m, u^f) / u^m \in [u_{\min}^m, u_{\max}^m], u^f \in [u_{\min}^f, u_{\max}^f] \right\}.$$

Under the collective model, no assumptions are made concerning the behaviour of household members should they fail to reach a cooperative agreement, nor are any restrictions imposed on the minimum utility payoffs that are acceptable to either household member under a joint agreement (aside from the minimum and maximum feasible utilities just defined). Nevertheless under the rationality postulate and given a sufficient degree of individual freedom on the part of household members to refuse certain outcomes, it seems reasonable to ask whether household members will indeed accept a cooperative outcome that provides them with less than some individually rational outcome, however that may be defined.

For example suppose we consider the *individual* constrained maximization problems for persons m and f under the worse possible scenario of a zero wage and zero nonwage income. Then it is easy to verify that the utility payoff to m under this scenario would be

$$V^m(p^0, p^1, 0, 0) = U^m(x^0(p^0, p^1, 0, 0), x^1(p^0, p^1, 0, 0), x^3(p^0, p^1, 0, 0)) = U^m(0, 0, T),$$

while the utility payoff to f would be

$$V^f(p^0, p^2, 0, 0) = U^f(x^0(p^0, p^2, 0, 0), x^2(p^0, p^2, 0, 0), x^4(p^0, p^2, 0, 0)) = U^f(0, 0, T),$$

regardless of the prices p^0 , p^1 and p^2 of the shared good and of the two consumption goods. However if $V^f(p^0, p^2, 0, 0) > u_{\min}^f$ then f would always do better by acting privately than she would in a cooperative agreement that offered less than $V^f(p^0, p^2, 0, 0)$. Similarly for m . In this example therefore as long as household members are always free to ‘go it alone’, it would seem reasonable, in the collective framework, to restrict the joint household to choose from the set $U^c \subset U$ given by

$$U^c \equiv \{(U^m, U^f) : x \in X, U^m \geq V^m(p^0, p^1, 0, 0), U^f \geq V^f(p^0, p^2, 0, 0)\}.$$

We wish to make two points concerning this argument. First, such a restriction of the joint household’s choice set in the collective framework would entail the assumption that household members were indeed willing and able to reject certain offers. Second, the utility payoffs $V^m(p^0, p^1, 0, 0)$ and $V^f(p^0, p^2, 0, 0)$ are by no means the only candidates for the individually rational payoffs in a collective model. Indeed any utility that would be available to a household member in his/her ‘next best alternative’ and that offers strictly more than the minimum feasible utility could be taken to be that household member’s individually rational payoff.

A Generalized Nash-Bargaining Model of Household Decisions

In this section we present a *generalized* Nash-bargaining model of intrahousehold allocations. This model is an extension of the symmetric Nash model of MHM that takes account of differences between and variations in the ‘bargaining strength’ of individual household members. For each h , let $\beta^h \in [0, 1]$ represent the weight attached to person h ’s utility gain in the Nash product function. β^h is taken to be a function of the exogenous parameters α , and $\sum_h \beta^h = 1$. We can interpret β^h as a measure of the bargaining strength of person h ; the higher β^h the greater the ‘influence’ of person h ’s preferences in the solution to the joint household problem. Let $\gamma = \frac{\beta^f}{\beta^m} \in [0, \infty)$ represent the utility weight of person f relative to that of person m . Finally, let $(V^m, V^f) \in U$

represent the disagreement utilities for m and f , defined by (2.1) and (2.2) respectively and which we interpret as the minimum utilities that m and f would be willing to accept under the joint household decision. Note that these are the same disagreement utilities used by MHM, Manser and Brown (1978, 1980) and Brown and Manser (1977, 1978) in their symmetric Nash-bargaining models of the household. Therefore for a given parameter vector α , the *generalized* Nash product function, $N(x'; \alpha)$, is given by $[U^m(x^m) - V^m] [U^f(x^f) - V^f]^{\gamma(\alpha)}$ and the joint decision problem becomes

$$\max_{x^i, \forall i} \left\{ [U^m(x^m) - V^m] [U^f(x^f) - V^f]^{\gamma(\alpha)} / y^m + y^f - p'q \geq 0 \right\}, \quad (\text{GN})$$

where V^m and V^f are given by expressions (2.1) and (2.2) respectively.¹³

Since, under both the symmetric and the generalized Nash models, household members are assumed to reject any allocation that offers less than the disagreement utilities, the effect of this assumption is to ensure that under a Nash cooperative agreement, household member h will receive no less than utility level V^h ($h = m, f$). Under the Nash models therefore, the joint household is effectively restricted to choose from the set $U^n \subset U$, given by

$$U^n = \{ (U^m, U^f) : x \in X, U^m \geq V^m, U^f \geq V^f \}.$$

Discussion

We have argued that under the symmetric and the generalized Nash-bargaining models the household is restricted to choosing from a subset, U^n , of the feasible utility allocations, U . We have additionally argued that under the collective framework there may be good reasons for also restricting the set of utility allocations from which a household can

¹³Note that we have transformed the usual Nash product function $[U^m(x^m) - V^m]^{\beta^m(\alpha)} [U^f(x^f) - V^f]^{\beta^f(\alpha)}$ by raising it to the power $\frac{1}{\beta^m(\alpha)}$. This represents a positive monotonic transformation and therefore does not alter the household's overall preference ordering nor the solution to the programme *GN*.

choose, say to a set such as U^c . Note however that the sets U^n and U^c could be defined more generally as

$$U^N \equiv \{ (U^m, U^f) : x \in X, U^m \geq u_n^m, U^f \geq u_n^f \}$$

and

$$U^C \equiv \{ (U^m, U^f) : x \in X, U^m \geq u_c^m, U^f \geq u_c^f \},$$

respectively, for some appropriately chosen (u_n^m, u_n^f) and (u_c^m, u_c^f) which belong to the interior of U . Under the symmetric Nash models of MHM, Manser and Brown (1978, 1980) and Brown and Manser (1977, 1978) and under the generalized version presented here, (u_n^m, u_n^f) is given by

$$(V^m(p^0, p^1, p^3, y^m), V^f(p^0, p^2, p^4, y^f)).$$

Under the collective model of CH we effectively have (u_c^m, u_c^f) given by (u_{\min}^m, u_{\min}^f) , although we also suggested that at the very least we might have (u_c^m, u_c^f) given by

$$(V^m(p^0, p^1, 0, 0), V^f(p^0, p^2, 0, 0)).$$

In fact there are any number of possibilities for choosing (u_n^m, u_n^f) and (u_c^m, u_c^f) . First, we could simply take them to be (u_{\min}^m, u_{\min}^f) as is already implied under the collective framework. Note that strictly speaking there is nothing in the Nash model to prevent the disagreement utilities also being given by (u_{\min}^m, u_{\min}^f) . However as argued earlier, if a household member's next best alternative offers more than his/her minimum feasible payoff, this begs the question as to why he/she would choose to accept an outcome, under the joint household agreement, that offered less than the outcome obtainable under some alternative arrangement. Second, if the utility payoffs from the worse possible private scenario, $V^m(p^0, p^1, 0, 0)$ and $V^f(p^0, p^2, 0, 0)$, were strictly greater than u_{\min}^m and u_{\min}^f , respectively, we could take (u_n^m, u_n^f) and (u_c^m, u_c^f) to be these worse case private outcomes.

Third, we could take (u_n^m, u_n^f) and (u_c^m, u_c^f) to be $(V^m(p^0, p^1, p^3, y^m), V^f(p^0, p^2, p^4, y^m))$ as is the case under existing models of Nash-bargained household behaviour. Finally, we might offer some alternative theory of household dissolution and allow the disagreement utilities, in any household bargaining problem, to be predicted by this theory.

The discussion in this section highlights the fact that however we choose to specify the allocations (u_n^m, u_n^f) and (u_c^m, u_c^f) , it is clear that once individual preferences and the exogenous parameters have been specified, the choice set of utility allocations (and the corresponding set of efficient allocations) facing a household will be identical under the generalized Nash and the collective models as long as we specify $(u_n^m, u_n^f) = (u_c^m, u_c^f)$.

2.3.2 A Formal Analysis in Utility Space

Before proceeding further we summarize the following results of the household bargaining problem specified in section 2.2 and the corresponding Pareto frontier specified in equation (2.5).¹⁴

Lemma 4. (i) X is convex. (ii) The set of feasible utilities, u^f , is a compact interval. (iii) $F'(u^f, \alpha) < 0$, $F''(u^f, \alpha) < 0$ and $F(u^f, \alpha)$ is continuously differentiable. (iv) For each u^f in its domain there is a unique consumption allocation $x \in X$ for which $U^f(x^f) = u^f$ and $U^m(x^m) = F(u^f, \alpha)$.

Proof. See Appendix. ■

Suppose we set the minimum acceptable utilities in the generalized Nash model equal to those in the collective model. Then $(u_n^m, u_n^f) = (u_c^m, u_c^f) = (\underline{u}^m, \underline{u}^f) \in \mathbf{U}$. The household's choice set is therefore

$$\mathbf{U}' \equiv \{(U^m, U^f) : x \in X, U^m \geq \underline{u}^m, U^f \geq \underline{u}^f\} \subset \mathbf{U},$$

¹⁴These results and their proof are analogous to those obtained by Thomas and Worrall (1988, p. 545) in a different (dynamic) context.

while the corresponding set of efficient allocations is given by

$$\mathbf{U}'_e = \{ (u^m, u^f) \in \mathbf{U}' : u^m = F(u^f, \alpha) \}.$$

Since $F' < 0$, the maximum utility for m is given by $\bar{u}^m = F(\underline{u}^f, \alpha)$ and the maximum utility for f is given by $\bar{u}^f = F^{-1}(\underline{u}^m, \alpha)$. We can now state the following definitions.¹⁵

Definition 1. Given a choice set, \mathbf{U}' , a joint household preference ordering, R , is said to rationalize a utility allocation $(u^m, u^f) \in \mathbf{U}'$ if and only if the allocation that would be chosen under R coincides with (u^m, u^f) , i.e.

$$\begin{aligned} & (u^m, u^f) \\ = & U^* \\ = & \{ (u^m, u^f) \in \mathbf{U}' : (u^m, u^f) R (\hat{u}^m, \hat{u}^f) \ \forall (\hat{u}^m, \hat{u}^f) \in \mathbf{U}' \}. \end{aligned}$$

Since \mathbf{U}' must be strictly convex, then under the appropriate concavity assumptions over R , the set U^* must be a singleton. In this case the preference ordering R strongly rationalizes (u^m, u^f) . If U^* is multi-valued, then R weakly rationalizes (u^m, u^f) if and only if $(u^m, u^f) \in U^*$. We extend definition 1 as follows.

Definition 2. Let the choice set, \mathbf{U}' , and the corresponding set of efficient allocations, \mathbf{U}'_e , be given. Let R^ω represent a class of household preference orderings such that for every value of the parameter $\omega \in [0, \infty)$, R^ω chooses a distinct utility allocation from the choice set \mathbf{U}' . A class of household preference orderings, R^ω , is said to rationalize the set of efficient utility allocations, \mathbf{U}'_e , if and only if for every $(F(u^f, \alpha), u^f) \in \mathbf{U}'_e$ there is a distinct value of ω such that the allocation that would be chosen under R^ω coincides

¹⁵Definition 1 adapts the definitions in Pollak (1977), on the weak and strong rationalization of a demand system by an individual's preference ordering, to the current context. Definition 2 extends definition 1 to a class of preference orderings.

with $(F(u^f, \alpha), u^f)$. Formally, $\forall (F(u^f, \alpha), u^f) \in U'_e$, there exists ω such that

$$\begin{aligned} & (F(u^f, \alpha), u^f) \\ = & U_e^* \\ = & \{ (F(u^f, \alpha), u^f) \in U'_e : (F(u^f, \alpha), u^f) R^\omega (F(\hat{u}^f, \alpha), \hat{u}^f) \ \forall (F(\hat{u}^f, \alpha), \hat{u}^f) \in U'_e \}. \end{aligned}$$

Let $(F(u^f, \alpha), u^f)$ and $(F(\tilde{u}^f, \alpha), \tilde{u}^f)$ be any two allocations in U'_e such that $(F(u^f, \alpha), u^f)$ is chosen under R^ω and $(F(\tilde{u}^f, \alpha), \tilde{u}^f)$ is chosen under $R^{\tilde{\omega}}$. Then

$$(F(u^f, \alpha), u^f) \neq (F(\tilde{u}^f, \alpha), \tilde{u}^f) \text{ if and only if } \omega \neq \tilde{\omega}.$$

As before since U' is strictly convex, then under the appropriate concavity assumptions over $R^\omega, \forall \omega$, the set U_e^* must also be a singleton.

We now present the main results of this section. Let $\gamma \in [0, \infty)$ represent the relative utility weight in the generalized Nash model GN with (V^m, V^f) replaced by $(\underline{u}^m, \underline{u}^f)$. We can now state the following result:

Proposition 1. Let the parameter vector, α , and the corresponding Pareto frontier, $F(\cdot, \alpha)$, be given. For every allocation $(F(u^f, \alpha), u^f)$ on the frontier, there exists a unique value of $\gamma(u^f, \alpha) \in [0, \infty)$ for which the allocation $(F(u^f, \alpha), u^f)$ would be chosen under the generalized Nash model. Suppose $(F(u^f, \alpha), u^f)$ and $(F(\tilde{u}^f, \alpha), \tilde{u}^f)$ are any two allocations on the Pareto frontier such that $(F(u^f, \alpha), u^f)$ is chosen when $\gamma = \gamma(u^f, \alpha)$ and $(F(\tilde{u}^f, \alpha), \tilde{u}^f)$ is chosen when $\gamma = \gamma(\tilde{u}^f, \alpha) = \tilde{\gamma}$. Then

$$(F(u^f, \alpha), u^f) \neq (F(\tilde{u}^f, \alpha), \tilde{u}^f) \text{ if and only if } \gamma(u^f, \alpha) \neq \gamma(\tilde{u}^f, \alpha).$$

Finally, as u^f rises from \underline{u}^f to its maximum value, \bar{u}^f , (and as $F(u^f, \alpha)$ correspondingly falls from \bar{u}^m to \underline{u}^m) $\gamma(u^f, \alpha)$ varies from 0 to ∞ .

Proof. The generalized Nash model GN can be equivalently specified, in utility space, as

$$\max_{U^m, U^f} \{ (U^m - \underline{u}^m) (U^f - \underline{u}^f)^\gamma / U^m = F(U^f, \alpha), U^m \geq \underline{u}^m, U^f \geq \underline{u}^f \},$$

or alternatively,

$$\max_{U^f} \{ (F(U^f, \alpha) - \underline{u}^m) (U^f - \underline{u}^f)^\gamma / U^m \geq \underline{u}^m, U^f \geq \underline{u}^f \}. \quad (\text{GN}_U)$$

Any solution to this problem must satisfy

$$F'(U^f, \alpha) (U^f - \underline{u}^f)^\gamma + (F(U^f, \alpha) - \underline{u}^m) \gamma (U^f - \underline{u}^f)^{\gamma-1} = 0$$

or

$$F'(U^f, \alpha) (U^f - \underline{u}^f)^\gamma + \frac{\gamma (F(U^f, \alpha) - \underline{u}^m) (U^f - \underline{u}^f)^\gamma}{(U^f - \underline{u}^f)} = 0.$$

Dividing through by $(U^f - \underline{u}^f)^\gamma$ implies

$$F'(U^f, \alpha) + \gamma \frac{F(U^f, \alpha) - \underline{u}^m}{U^f - \underline{u}^f} = 0,$$

and re-arranging gives us

$$\gamma = \gamma(U^f, \alpha) = -F'(U^f, \alpha) \frac{U^f - \underline{u}^f}{F(U^f, \alpha) - \underline{u}^m}. \quad (2.8)$$

For every value of U^f and corresponding utility allocation $(F(U^f, \alpha), U^f)$, expression (2.8) tells us the value of γ for which the allocation $(F(U^f, \alpha), U^f)$ would be an optimal solution to GN_U . Alternatively, if the value of γ is determined exogenously, then (2.8) tells us the utility allocation $(F(U^f, \alpha), U^f)$ that represents an optimal solution to GN_U . Furthermore, since

$$\frac{\partial \gamma(U^f, \alpha)}{\partial U^f} = -F'(U^f, \alpha) \frac{(F(U^f, \alpha) - \underline{u}^m) - (U^f - \underline{u}^f) F'(U^f, \alpha)}{(F(U^f, \alpha) - \underline{u}^m)^2}$$

$$-F''(U^f, \alpha) \frac{U^f - \underline{u}^f}{F(U^f, \alpha) - \underline{u}^m}$$

is always strictly positive, then $\gamma(U^f, \alpha)$ must be a positive monotonic function of U^f . This implies that for any two allocations $(F(u^f, \alpha), u^f)$ and $(F(\tilde{u}^f, \alpha), \tilde{u}^f)$ on the Pareto frontier, we must have $(F(u^f, \alpha), u^f) \neq (F(\tilde{u}^f, \alpha), \tilde{u}^f)$ if and only if $\gamma(u^f, \alpha) \neq \gamma(\tilde{u}^f, \alpha)$. Therefore for every allocation $(F(U^f, \alpha), U^f)$ there is a *distinct* value of $\gamma(U^f, \alpha)$ for which $(F(U^f, \alpha), U^f)$ would be chosen under GN_U .

Finally, it is easy to verify that whenever $U^f = \underline{u}^f$, $\gamma(U^f, \alpha) = 0$. Also, as U^f rises towards \bar{u}^f , $F(U^f, \alpha)$ approaches \underline{u}^m and $F'(U^f, \alpha)$ becomes more negative. Therefore $\gamma(U^f, \alpha)$ approaches ∞ . ■

We mentioned in section 2.1 that in CH, the collective decision process is sometimes formally represented as the constrained maximization of a weighted sum of individual utilities, with the weights taken to be a function of the exogenous parameters of the household's decision problem. Under this interpretation household decisions are a solution to

$$\max_{x^i, \forall i} \{U^m(x^m) + \mu(\alpha) U^f(x^f) / y^m + y^f - p'q \geq 0\}, \quad (\text{C2})$$

where $\mu(\alpha) \in [0, \infty)$. Here μ is interpreted as the weight of person f 's utility in the household's decision, relative to that of person m . The next result shows that for every efficient utility allocation, there is a distinct value of $\mu \in [0, \infty)$ for which the given allocation will be a solution under the sum of utilities interpretation of the collective model.

Proposition 2. As before, let α and the corresponding frontier $F(\cdot, \alpha)$ be given. For every allocation $(F(u^f, \alpha), u^f)$ on the frontier, there exists a unique value $\mu(u^f, \alpha) \in [0, \infty)$ for which $(F(u^f, \alpha), u^f)$ would be chosen under the sum of utilities collective model. Suppose $(F(u^f, \alpha), u^f)$ and $(F(\tilde{u}^f, \alpha), \tilde{u}^f)$ are any two allocations on the frontier such that $(F(u^f, \alpha), u^f)$ is chosen when $\mu = \mu(u^f, \alpha)$ and $(F(\tilde{u}^f, \alpha), \tilde{u}^f)$ is chosen

when $\mu = \mu(\tilde{u}^f, \alpha) = \tilde{\mu}$. Then

$$(F(u^f, \alpha), u^f) \neq (F(\tilde{u}^f, \alpha), \tilde{u}^f) \text{ if and only if } \mu(u^f, \alpha) \neq \mu(\tilde{u}^f, \alpha).$$

Finally, as u^f rises from \underline{u}^f to its maximum value, \bar{u}^f , (and $F(u^f, \alpha)$ correspondingly falls from \bar{u}^m to \underline{u}^m) $\mu(u^f, \alpha)$ varies from 0 to ∞ .

Proof. The sum of utilities collective model C2 can be equivalently expressed, in utility space, as

$$\max_{U^m, U^f} \{U^m + \mu U^f / U^m = F(U^f, \alpha), U^m \geq \underline{u}^m, U^f \geq \underline{u}^f\},$$

or equivalently,

$$\max_{U^f} \{F(U^f, \alpha) + \mu U^f / U^m \geq \underline{u}^m, U^f \geq \underline{u}^f\}. \quad (\text{C2}_U)$$

It is easy to verify that any solution to this problem must satisfy

$$\mu = \mu(U^f, \alpha) = -F'(U^f, \alpha). \quad (2.9)$$

For every value of U^f and corresponding utility allocation $(F(U^f, \alpha), U^f)$, expression (2.9) tells us the value of μ for which $(F(U^f, \alpha), U^f)$ would be an optimal solution to C2_U . Alternatively, if the value of μ is determined exogenously, then (2.9) tells us the utility allocation $(F(U^f, \alpha), U^f)$ that represents an optimal solution to (2.9).

Furthermore, since $\frac{\partial \mu(u^f, \alpha)}{\partial U^f} = -F''(U^f, \alpha)$ is always strictly positive, $\mu(u^f, \alpha)$ must be a positive monotonic function of U^f . This implies that for any two allocations $(F(u^f, \alpha), u^f)$ and $(F(\tilde{u}^f, \alpha), \tilde{u}^f)$ on the Pareto frontier, we must have $(F(u^f, \alpha), u^f) \neq (F(\tilde{u}^f, \alpha), \tilde{u}^f)$ if and only if $\mu(u^f, \alpha) \neq \mu(\tilde{u}^f, \alpha)$. Therefore for every allocation $(F(U^f, \alpha), U^f)$ there is a *distinct* value of $\mu(U^f, \alpha)$ for which $(F(U^f, \alpha), U^f)$ would be chosen under C2_U .

Finally, as U^f rises towards \bar{u}^f , $\mu(u^f, \alpha)$ approaches ∞ . ■

Propositions 1 and 2 support our conjecture that, given individual preferences, a

particular cardinal representation of those preferences and a set of exogenous parameters, the generalized Nash and the collective classes of household preference orderings will rationalize the same set of efficient household utility allocations. Combining propositions 1 and 2 gives us the following corollary.

Corollary 1. Suppose the value of μ in the collective model is some function of the parameter vector α , with the mapping $\mu(\alpha)$ being positive and differentiable. Then we can find a positive, differentiable mapping $\gamma(\mu(\alpha), \alpha)$ such that, for every parameter vector, α , and for every positive, differentiable mapping, $\mu(\alpha)$, the allocation chosen under the collective model with relative utility weight $\mu(\alpha)$ coincides with the allocation chosen under the generalized Nash model with relative utility weight $\gamma(\mu(\alpha), \alpha)$.

Proof. Choose any positive, differentiable mapping $\mu(\alpha)$ from the parameter space into $[0, \infty)$ and pick a particular parameter vector $\hat{\alpha}$ from the set of all possible parameter vectors. Let the resulting value $\hat{\mu} = \mu(\hat{\alpha})$ be the relative utility weight in the collective model and let $F(\cdot, \hat{\alpha})$ represent the Pareto frontier corresponding to the parameter vector $\hat{\alpha}$. Proposition 2 demonstrated that the collective model will select, as a solution, the allocation $(F(U^f, \hat{\alpha}), U^f)$ that satisfies $\hat{\mu} = -F'(U^f, \hat{\alpha})$. Denote the chosen allocation by $(F(U^{f*}, \hat{\alpha}), U^{f*})$. Then $\hat{\mu} = -F'(U^{f*}, \hat{\alpha})$. From proposition 1 we know that we can find a value $\gamma \in [0, \infty)$ for which the allocation $(F(U^{f*}, \hat{\alpha}), U^{f*})$ would also be chosen under a generalized Nash model with relative utility weight γ . The required value of γ , denoted by $\hat{\gamma}$, is given by

$$\hat{\gamma} = -F'(U^{f*}, \hat{\alpha}) \frac{U^{f*} - \underline{u}^f}{F(U^{f*}, \hat{\alpha}) - \underline{u}^m}.$$

Repeating this procedure for every possible parameter vector α allows us to define the positive, differentiable mapping

$$\gamma(\mu(\alpha), \alpha) = -F'(U^f, \alpha) \frac{U^f - \underline{u}^f}{F(U^f, \alpha) - \underline{u}^m}, \quad (2.10)$$

with U^f on the right hand side chosen according to

$$\mu(\alpha) = -F'(U^f, \alpha), \quad (2.11)$$

for any positive, differentiable mapping $\mu(\alpha)$. Let $G(\cdot, \alpha) = F'^{-1}(\cdot, \alpha), \forall \alpha$, and $z = -\mu(\alpha)$. We can therefore express (2.11) equivalently as

$$\begin{aligned} U^f &= F'^{-1}(-\mu(\alpha), \alpha) \\ &= G(z, \alpha), \quad z = -\mu(\alpha), \end{aligned} \quad (2.12)$$

and the mapping $\gamma(\mu(\alpha), \alpha)$ can thus be expressed entirely in terms of α and $\mu(\alpha)$:

$$\gamma(\mu(\alpha), \alpha) = -F'(G(z, \alpha), \alpha) \frac{G(z, \alpha) - \underline{u}^f}{F(G(z, \alpha), \alpha) - \underline{u}^m}, \quad (2.13)$$

where $z = -\mu(\alpha)$. ■

We can also state the following result.

Corollary 2. For *any* positive, differentiable mapping $\mu(\alpha)$ representing the relative utility weights in the collective model, if the relative utility weights in the generalized Nash model are chosen according to the mapping $\gamma(\mu(\alpha), \alpha)$, as specified in (2.10) or (2.13), then both models will always yield the same solution for any given parameter vector α .

Proof. Choose any positive, differentiable mapping $\mu(\alpha)$ and set $\gamma(U^f, \alpha) = \gamma(\mu(\alpha), \alpha)$ on the left hand side of expression (2.8). Then on the left hand side of (2.8) we must have U^f chosen according to (2.11). But this is just the solution to the collective model. It follows that U^f on the right hand side of (2.8), i.e. the solution to the generalized Nash model, must also satisfy (2.11), the solution to the collective model. ■

Suppose the parameter vector α is given so that the Pareto frontier is fixed. Then for any value of μ (obtained under alternative mappings $\mu(\alpha)$) and corresponding utility

allocation $(F(U^f, \alpha), U^f)$ determined by (2.11) or (2.12), expressions (2.10) and (2.13) can be interpreted as telling us the value of γ under which the generalized Nash model will yield the same utility allocation. On the other hand, suppose the parameter vector α is allowed to vary for a given mapping $\mu(\alpha)$. Again expressions (2.10) and (2.13) can be interpreted as telling us the value of γ under which the generalized Nash model will yield the same utility allocation.

Corollary 2 demonstrates that if the generalized Nash relative utility weights are chosen according to $\gamma(\mu(\alpha), \alpha)$, as specified in (2.10) or (2.13), then both the generalized Nash and the collective models will always choose the same utility allocation along the Pareto frontier. It follows that the *total effect* of a change in the parameter $w \in \alpha$ on the chosen utility allocation must also be the same under both models. Note, however, that our result on the equivalence of the generalized Nash and the collective frameworks does not hinge on the relative utility weight $\gamma(\mu(\alpha), \alpha)$ being used in the generalized Nash model whenever the relative weight $\mu(\alpha)$ is used in the collective. By arguing that the two approaches are equivalent we are merely saying that both approaches rationalize the same set of efficient utility allocations whenever they specify the same minimum utility payoffs. Both approaches therefore generate the same continuum of efficient household utility allocations, and neither approach restricts the household *a priori* to a particular location on the Pareto frontier.

2.4 Properties of Collective and Generalized Nash-Bargained Household Demand Systems

We now investigate the properties of the household demand systems implied by the collective and the generalized Nash frameworks. Suppose both approaches make the same specifications with regard to each person's minimum utility payoffs. Then the resulting generalized Nash and collective demand systems will exhibit the same properties. This will be true regardless of *how* the minimum utilities are specified. In this regard there

are essentially three alternatives: either no minimum payoffs are specified so that the household chooses from the entire set of feasible utilities; or a pair of minimum utilities, different from the minimum feasible utilities, are specified for each player and are assumed to be constant; or a pair of disagreement utilities are specified which depend on certain parameters of the household decision problem. The point here is that it does not matter what assumptions we make regarding these minimum utilities. As long as both models make the same assumptions, then both approaches will yield demand systems that exhibit the same properties.

Under the symmetric Nash-bargaining approach developed in MHM and Manser and Brown (1980) (and the generalized version presented below), any cooperative outcome must offer, to each individual, a utility payoff that is no less than that individual's 'opportunity cost' of household membership as represented by their disagreement utility. In these models each person's disagreement utility is given by the maximized value of their individual utility in a private constrained maximization problem and therefore depends upon the parameters of this private optimization problem. On the other hand, under the collective approach offered by CH the household chooses from among the entire set of feasible utilities.

In the remainder of this chapter we demonstrate that this collective approach entails greater restrictions on household demands than is entailed by a generalized Nash approach that takes account of the opportunity cost of household membership and of how this opportunity cost is related to the parameters of the household decision problem. Section 2.4.1 derives the fundamental matrix equations of comparative statics for the collective model and derives the structural properties of the collective system of household demands. Section 2.4.2 carries out a similar analysis for the generalized Nash model. In section 2.5 we discuss the relationship between the collective and the generalized Nash approaches and demonstrate that the collective model presented in CH entails more restrictions on household demand systems than the generalized Nash approach presented below. We also summarize the comparative statics of the traditional and the symmetric Nash demand

systems and demonstrate a nesting structure between all four models.

2.4.1 Collective Demand Systems and Comparative Statics

For any parameter vector α and a corresponding value $\mu(\alpha) \in [0, \infty)$, collective household decisions must be the solution to

$$\max_{x^i, \forall i} \{U^m(x^m) + \mu(\alpha)U^f(x^f) / y^m + y^f - p'q \geq 0\}. \quad (\text{C2})$$

The set of feasible consumption allocations, X , is convex (lemma 4), closed (since all feasible allocations must satisfy the budget constraint), bounded from below in the x 's and contains the null vector (assumption 1). Also, under assumption 2, the collective household objective function, $U(x'; \alpha)$, given by

$$U(x'; \alpha) = U^m(x^m) + \mu(\alpha)U^f(x^f),$$

is twice continuously differentiable and strictly concave on X , as well as increasing in each x^i . Therefore the solution to C2 exists and is unique. If, in the solution to C2, we have $x^{i*} = x^{i*}(\alpha, \mu(\alpha)) > 0, \forall i$, (an interior solution) and $\lambda^* = \lambda^*(\alpha, \mu(\alpha)) > 0$, where λ is the Lagrange multiplier on the joint household budget constraint in the problem C2, then the first order conditions for this programme can be written as¹⁶

¹⁶Note that combining the first order conditions for $x^i, i = 1, \dots, 4$, gives us

$$p^3 U_{x^1}^m(x^m) = p^1 U_{x^3}^m(x^m) \quad (2.14)$$

and

$$p^4 U_{x^2}^f(x^f) = p^2 U_{x^4}^f(x^f) \quad (2.15)$$

which are the conditions obtained in CH (p. 82) for an efficient allocation of the private goods. Also, the condition for x^0 implies $\frac{1}{\lambda} U_{x^0}^m(x^m) + \frac{\mu(\alpha)}{\lambda} U_{x^0}^f(x^f) = p^0$, where $\frac{1}{\lambda} = \frac{p^i}{U_{x^i}^m(x^m)}, i = 1, 3$, and $\frac{1}{\lambda} = \frac{p^j}{\mu(\alpha)U_{x^j}^f(x^f)}, j = 2, 4$. Therefore we must have

$$\frac{U_{x^0}^m(x^m)}{U_{x^i}^m(x^m)/p^i} + \frac{U_{x^0}^f(x^f)}{U_{x^j}^f(x^f)/p^j} = p^0, \forall i = 1, 3, j = 2, 4. \quad (2.16)$$

$$U_{x^i}^m(x^m) + \mu(\alpha) U_{x^i}^f(x^f) - \lambda p^i = 0, \forall i, \quad (2.17)$$

and

$$y^m + y^f - p'q = 0. \quad (2.18)$$

To obtain the fundamental equations of comparative statics, totally differentiate conditions (2.17) and (2.18) with respect to each of the parameters in α . (See the Appendix to this chapter for a complete derivation). The resulting equations can be arranged into a system of matrix equations, evaluated at the solution $x^{i*} = x^{i*}(\alpha, \mu(\alpha))$, $\forall i$, and $\lambda^* = \lambda^*(\alpha, \mu(\alpha))$. These are given by:

$$\begin{bmatrix} U_{xx} & -p \\ (5 \times 5) & (5 \times 1) \end{bmatrix} \begin{bmatrix} X_p & X_y & X_\mu \\ (5 \times 5) & (5 \times 2) & (5 \times 1) \end{bmatrix} + \begin{bmatrix} -\lambda I + U_{xp} & U_{xy} & U_{x\mu} \\ (5 \times 5) & (5 \times 2) & (5 \times 1) \end{bmatrix} = \begin{bmatrix} 0 \\ (6 \times 8) \end{bmatrix}. \quad (2.19)$$

To understand the members of (2.19), first define U_x as an array of the gradients of each person's utility function,

$$U_x \equiv \begin{bmatrix} \frac{\partial U^f}{\partial x^i} : \frac{\partial U^m}{\partial x^i} \\ (5 \times 2) & (5 \times 1) \end{bmatrix};$$

μ_p as a (1×5) vector of the partial derivatives of μ with respect to the prices,

$$\mu_p \equiv \begin{bmatrix} \frac{\partial \mu}{\partial p^k} \\ (1 \times 5) \end{bmatrix};$$

and μ_y as a (1×2) vector of the partial derivatives of μ with respect to the nonwage

(2.16) is the standard condition for an efficient level of provision of a public good, such as x^0 , and is implied under the collective framework when we include shared goods in the consumption baskets of f and m .

incomes,

$$\mu_y \equiv \left[\frac{\partial \mu}{\partial y^m} ; \frac{\partial \mu}{\partial y^f} \right]_{(1 \times 2)}.$$

Then the members of (2.19) are given by U_{xx} , the (5×5) Hessian of the collective objective function $U(x'; \alpha)$,

$$U_{xx} \equiv \left[\frac{\partial^2 U^m}{\partial x^i \partial x^j} + \mu(\alpha) \frac{\partial^2 U^f}{\partial x^i \partial x^j} \right]_{(5 \times 5)};$$

$$U_{xp} \equiv \frac{\partial U^f}{\partial x^i} \mu_p;_{(5 \times 5) \quad (5 \times 1) \quad (1 \times 5)}$$

$$U_{xy} \equiv \frac{\partial U^f}{\partial x^i} \mu_y;_{(5 \times 2) \quad (5 \times 1) \quad (1 \times 2)}$$

and $U_{x\mu}$, a (5×1) vector of the partial derivatives of $U^f(x^f)$,

$$U_{x\mu} \equiv \left[\frac{\partial U^f}{\partial x^i} \right]_{(5 \times 1)}.$$

The matrices p and q are given by $p = [p^i]_{(5 \times 1)}$, $p' = [p^j]_{(1 \times 5)}$ and $q' = [q^{k*}]_{(1 \times 5)}$; i' is a (1×2) vector of ones; $\lambda I_{(5 \times 5)}$ is the (5×5) identity matrix multiplied by the Lagrange multiplier λ^* ; X_p is the (5×5) matrix of uncompensated price effects,

$$X_p \equiv \left[\frac{\partial x^{j*}}{\partial p^k} \right]_{(5 \times 5)};$$

X_y is the (5×2) matrix of the marginal impacts of nonwage incomes on consumption demands,

$$X_y \equiv \left[\frac{\partial x^{j*}}{\partial y^m} ; \frac{\partial x^{j*}}{\partial y^f} \right]_{(5 \times 2)};$$

and X_μ is a (5×1) vector of the effect of μ on consumption demands,

$$X_\mu \equiv \left[\frac{\partial x^{j*}}{\partial \mu} \right]_{(5 \times 1)}.$$

We also have λ_p , λ_y and λ_μ given, respectively, by $\left[\frac{\partial \lambda^*}{\partial p^k} \right]$, $\left[\frac{\partial \lambda^*}{\partial y^m} : \frac{\partial \lambda^*}{\partial y^j} \right]$ and $\left[\frac{\partial \lambda^*}{\partial \mu} \right]$. Finally, 0 and 0 are, respectively, the scalar zero and (6×8) matrix of zeros.

We are now in a position to examine the structural properties of observed demand functions as implied by the collective model. Given a parameter vector α and a corresponding value $\mu(\alpha) \in [0, \infty)$, the solution to the programme C2 is a system of collective demand equations $x^{i*} = x^{i*}(\alpha, \mu(\alpha))$, $\forall i$, as well as a solution for $\lambda^* = \lambda^*(\alpha, \mu(\alpha))$. These collective demands describe the response of household demands not only to variations in prices and nonwage incomes but also to variations in the value of μ , i.e. to changes in the internal ‘distribution of bargaining power’. However as Browning and Chiappori (1996) argue, we do not in fact observe variations in the value of μ nor do we observe the response of household demands to such variation. Instead we observe some demand system $x^i(\alpha)$, $\forall i$, defined by $x^i(\alpha) = x^{i*}(\alpha, \mu(\alpha))$. We also observe the response of these demands to variations in prices and nonwage incomes, *for some specific, unobservable mapping* $\mu(\alpha)$ that characterizes the distribution of bargaining power within the household in question. We are interested in obtaining the structural properties, implied by the collective framework, of such observable demands. First we make the following assumption:

Assumption 3. The rank of $p' = 1$.

Since the collective household objective function, $U(x; \alpha)$, is twice continuously differentiable and strictly concave on X , it follows that the Hessian of $U(x; \alpha)$, the (5×5) matrix $U_{xx} = \left[\frac{\partial^2 U}{\partial x^i \partial x^j} \right]_{(5 \times 5)}$, must be symmetric and negative definite. Since U_{xx} is symmetric

and p' is the transpose of p , then the bordered Hessian $\begin{bmatrix} U_{xx} & -p \\ -p' & 0 \end{bmatrix}$ is also symmetric.

Therefore, since U_{xx} is negative definite and, under assumption 3, the rank of $p' = 1$,

we can apply the Caratheodory-Samuelson theorem¹⁷ to see that $\begin{bmatrix} U_{xx} & -p \\ -p' & 0 \end{bmatrix}^{-1}$ exists

and can be partitioned as $\begin{bmatrix} a_c & g_c \\ g_c' & k_c \end{bmatrix}$, where a_c is a symmetric and negative semi definite matrix.

We are now in a position to re-arrange the matrix equations (2.19) to obtain

$$\begin{bmatrix} X_p & X_y & X_\mu \\ \lambda_p & \lambda_y & \lambda_\mu \end{bmatrix} = - \begin{bmatrix} a_c & g_c \\ g_c' & k_c \end{bmatrix} \begin{bmatrix} -\lambda I + U_{xp} & U_{xy} & U_{x\mu} \\ -q' & i' & 0 \end{bmatrix}. \quad (2.20)$$

From (2.20) we see that

$$X_y = - \frac{a_c}{(5 \times 5)} \frac{U_{xy}}{(5 \times 2)} - \frac{g_c}{(5 \times 1)} \frac{i'}{(1 \times 2)}.$$

Post-multiplying by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ implies

$$\begin{aligned} X_{ym} - X_{yf} &= - \frac{a_c}{(5 \times 5)} \frac{U_{xy}}{(5 \times 2)} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= - \frac{a_c}{(5 \times 5)} \frac{U_{x\mu}}{(5 \times 1)} \frac{\mu_y}{(1 \times 2)} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

¹⁷See Takayama, A. (1994, pp. 130). See also Caratheodory, C. (1967, pp. 195-96) and Samuelson, P. A. (1947, pp. 378-79).

$$= \begin{matrix} X_\mu & \mu_y \\ (5 \times 1) & (1 \times 2) \end{matrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \quad (2.21)$$

Condition (2.21) tells us that under the collective framework, as long as changes in y_m and y_f have different effects on the value of μ , then $X_{y_m} - X_{y_f} \neq 0$ and there will be no income-pooling under the collective model. In this case each family member's nonwage income will enter the collective demand equations separately.

Now solving for $\begin{matrix} g_c \\ (5 \times 1) \end{matrix}$ implies

$$\begin{matrix} g_c \\ (5 \times 1) \end{matrix} = -\frac{1}{2} \begin{bmatrix} X_y + \begin{matrix} a_c \\ (5 \times 5) \end{matrix} \frac{\partial U^f}{\partial x^i} \begin{matrix} \mu_y \\ (1 \times 2) \end{matrix} \end{bmatrix} \begin{matrix} i \\ (2 \times 1) \end{matrix}.$$

From (2.20) we also have

$$\begin{matrix} X_p \\ (5 \times 5) \end{matrix} = - \begin{matrix} a_c \\ (5 \times 5) \end{matrix} \begin{bmatrix} -\lambda I + \frac{\partial U^f}{\partial x^i} \begin{matrix} \mu_p \\ (1 \times 5) \end{matrix} \end{bmatrix} + \begin{matrix} g_c & q' \\ (5 \times 1) & (1 \times 5) \end{matrix}.$$

Replacing $\begin{matrix} g_c \\ (5 \times 1) \end{matrix}$ in this expression and re-arranging gives us

$$\begin{aligned} \begin{matrix} X_p \\ (5 \times 5) \end{matrix} + \frac{1}{2} \begin{matrix} X_y & i & q' \\ (5 \times 2) & (2 \times 1) & (1 \times 5) \end{matrix} &= \begin{matrix} a_c \\ (5 \times 5) \end{matrix} \frac{\lambda I}{(5 \times 5)} - \begin{matrix} a_c \\ (5 \times 5) \end{matrix} \frac{\partial U^f}{\partial x^i} \begin{matrix} \mu_p \\ (1 \times 5) \end{matrix} + \frac{1}{2} \begin{matrix} \mu_y & i & q' \\ (1 \times 2) & (2 \times 1) & (1 \times 5) \end{matrix} \\ &= \begin{matrix} a_c \\ (5 \times 5) \end{matrix} \frac{\lambda I}{(5 \times 5)} + \begin{matrix} X_\mu \\ (5 \times 1) \end{matrix} \begin{matrix} \mu_p & i & q' \\ (1 \times 5) & (2 \times 1) & (1 \times 5) \end{matrix}. \end{aligned} \quad (2.22)$$

We follow Browning and Chiappori (1996) and interpret (2.22) as follows. If $\frac{\partial x^{j*}}{\partial y^m} = \frac{\partial x^{j*}}{\partial y^f}$, $\forall j$, then the left hand side of (2.22) reduces to $\begin{matrix} X_p \\ (5 \times 5) \end{matrix} + \begin{matrix} X_Y & q' \\ (5 \times 1) & (1 \times 5) \end{matrix}$, where $X_Y \equiv \begin{bmatrix} \frac{\partial x^{j*}}{\partial Y} \end{bmatrix}$ and $Y = y^m + y^f$. This is the usual Slutsky matrix, with Y representing aggregate household income. If $\frac{\partial x^{j*}}{\partial y^m} \neq \frac{\partial x^{j*}}{\partial y^f}$, $\forall j$, the left hand side of expression (2.22) represents the Slutsky matrix, generalized to take account of possible differences in the response

of household demands to variations in individual income. Suppose the price of good k increases by dp^k , with the change compensated by an increase in household income of $q^{k*}dp^k$. The effect of this on good j is

$$\frac{dx^{j*}}{dp^k} = \frac{\partial x^{j*}}{\partial p^k} + \frac{1}{2} \left(\frac{\partial x^{j*}}{\partial y^m} + \frac{\partial x^{j*}}{\partial y^f} \right) q^{k*}, \forall j, k,$$

which represents the elements in the Slutsky matrix on the left hand side of (2.22). Condition (2.22) therefore tells us that this effect consists of two components: a substitution effect, given by the corresponding term in the symmetric and negative semi definite matrix, $a_c \lambda I$, and which holds household utility and the relative utility weight constant; and an effect on the value of μ , given by

$$\frac{d\mu}{dp^k} = \frac{\partial \mu}{\partial p^k} + \frac{1}{2} \left(\frac{\partial \mu}{\partial y^m} + \frac{\partial \mu}{\partial y^f} \right) q^{k*}, \forall k$$

which in turn will affect the consumption of good j by an amount

$$\frac{dx^{j*}}{d\mu} = \frac{\partial x^{j*}}{\partial \mu}, \forall j.$$

We can now summarize the main restrictions on any system of household demands implied by the collective framework as follows:

1. The Slutsky matrix $X_p + \frac{1}{2} X_y \begin{matrix} i & q' \\ (5 \times 2) & (2 \times 1) & (1 \times 5) \end{matrix}$ need not be symmetric and negative semi definite.
2. The Slutsky matrix is decomposed according to

$$X_p + \frac{1}{2} X_y \begin{matrix} i & q' \\ (5 \times 2) & (2 \times 1) & (1 \times 5) \end{matrix} = a_c \lambda I + X_\mu \left\{ \begin{matrix} \mu_p & + \frac{1}{2} \mu_y \begin{matrix} i & q' \\ (1 \times 2) & (2 \times 1) & (1 \times 5) \end{matrix} \end{matrix} \right\}. \quad (2.22)$$

3. As long as $\frac{\partial \mu}{\partial y^m} \neq \frac{\partial \mu}{\partial y^f}$, collective decision making implies the absence of income

pooling within the household, i.e.

$$\begin{matrix} X_{y_m} & - & X_{y_f} & = & X_{\mu} & \mu_y \\ (5 \times 1) & & (5 \times 1) & & (5 \times 1)(1 \times 2) \end{matrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \neq 0. \quad (2.21)$$

This completes our analysis of the restrictions on household demand systems implied by the collective model of household decision making. We now turn to an analysis of the generalized Nash-bargaining model.

2.4.2 Generalized Nash-Bargained Demand Systems and Comparative Statics

For any parameter vector α and a corresponding value for $\gamma(\alpha) = \frac{\beta^f(\alpha)}{\beta^m(\alpha)} \in [0, \infty)$, any system of Nash-bargained household demands must be a solution to

$$\max_{x^i, \forall i} \left\{ [U^m(x^m) - V^m] [U^f(x^f) - V^f]^{\gamma(\alpha)} / y^m + y^f - p'q \geq 0 \right\}, \quad (\text{GN})$$

where the disagreement utilities are given by

$$\begin{aligned} V^m &= V^m(p^0, p^1, p^3, y^m) \\ &= \max_{x^i, i=0,1,3} \{ U^m(x^m) / p^0 x_m^0 + p^1 x^1 + p^3 x^3 \leq y^m + p^3 T \} \end{aligned}$$

and

$$\begin{aligned} V^f &= V^f(p^0, p^2, p^4, y^f) \\ &= \max_{x^i, i=0,2,4} \{ U^f(x^f) / p^0 x_f^0 + p^2 x^2 + p^4 x^4 \leq y^f + p^4 T \}. \end{aligned}$$

Under assumption 2, the generalized Nash product function,

$$N(x'; \alpha) = [U^m(x^m) - V^m] [U^f(x^f) - V^f]^{\gamma(\alpha)},$$

is twice continuously differentiable on X . Since $\gamma(\alpha) \in [0, \infty)$, $N(x'; \alpha)$ is increasing in the x 's. Furthermore since $\beta^h(\alpha) \in [0, 1]$, $\forall h$, and $\sum_h \beta^h(\alpha) = 1$, $N(x'; \alpha)$ is strictly quasi-concave (and hence quasi-concave) on U . Therefore since the set X is convex, closed, bounded from below in the x 's and contains the null vector, the solution to GN exists and is unique.

Suppose, in the solution to GN, we have

$$\hat{x}^i = \hat{x}^i(\alpha, \gamma(\alpha), V^m(p^0, p^1, p^3, y^m), V^f(p^0, p^2, p^4, y^f)) > 0, \forall i,$$

(an interior solution) and

$$\hat{\lambda} = \hat{\lambda}(\alpha, \gamma(\alpha), V^m(p^0, p^1, p^3, y^m), V^f(p^0, p^2, p^4, y^f)) > 0,$$

where λ is the Lagrange multiplier on the joint household budget constraint in the problem GN. Then by differentiating the appropriate Lagrangian we obtain the first order conditions¹⁸

¹⁸Since conditions (2.23) imply

$$\begin{aligned} [U^f(x^f) - V^f]^{\gamma(\alpha)} U_{x^0}^m(x^m) + \gamma(\alpha) \frac{[U^m(x^m) - V^m]}{[U^f(x^f) - V^f]} [U^f(x^f) - V^f]^{\gamma(\alpha)} U_{x^0}^f(x^f) &= \lambda(\alpha) p^0, \\ [U^f(x^f) - V^f]^{\gamma(\alpha)} U_{x^i}^m(x^m) &= \lambda(\alpha) p^i, i = 1, 3, \\ &\text{and} \\ \gamma(\alpha) \frac{[U^m(x^m) - V^m]}{[U^f(x^f) - V^f]} [U^f(x^f) - V^f]^{\gamma(\alpha)} U_{x^i}^f(x^f) &= \lambda(\alpha) p^i, i = 2, 4, \end{aligned}$$

then combining the conditions on $x^i, i = 1, \dots, 4$, gives us the same conditions as (2.14) and (2.15), shown in CH to characterize an efficient allocation of the private consumption goods. Conditions (2.23) also imply $\frac{1}{\lambda} U_{x^0}^m(x^m) + \frac{1}{\lambda} \gamma(\alpha) \frac{[U^m(x^m) - V^m]}{[U^f(x^f) - V^f]} U_{x^0}^f(x^f) = p^0$, where $\frac{1}{\lambda} = \frac{p^i}{U_{x^i}^m(x^m)}, i = 1, 3$, and $\frac{1}{\lambda} = \frac{p^j}{\gamma(\alpha) \frac{[U^m(x^m) - V^m]}{[U^f(x^f) - V^f]} U_{x^j}^f(x^f)}, j = 2, 4$. We therefore have

$$\frac{U_{x^0}^m(x^m)}{U_{x^i}^m(x^m)/p^i} + \frac{U_{x^0}^f(x^f)}{U_{x^j}^f(x^f)/p^j} = p^0, \forall i = 1, 3, j = 2, 4,$$

which is the same condition as (2.16) and which characterizes an efficient level of the shared good x^0 . As expected, the generalized Nash model of household decisions leads to an efficient allocation, within the household, of both the private consumption and leisure goods and of the shared good.

$$\begin{aligned}
& [U^f(x^f) - V^f]^{\gamma(\alpha)} U_{xi}^m(x^m) \\
& + \gamma(\alpha) \frac{[U^m(x^m) - V^m]}{[U^f(x^f) - V^f]} [U^f(x^f) - V^f]^{\gamma(\alpha)} U_{xi}^f(x^f) - \lambda p^i \\
& = 0, \forall i,
\end{aligned} \tag{2.23}$$

and

$$y^m + y^f - p'q = 0. \tag{2.24}$$

To obtain the fundamental equations of comparative statics, totally differentiate conditions (2.23) and (2.24) with respect to each of the parameters in α . (Again see the Appendix to this chapter for a complete derivation). We can arrange the resulting equations into a system of matrix equations, evaluated at the solution

$$\hat{x}^i = \hat{x}^i(\alpha, \gamma(\alpha), V^m(p^0, p^1, p^3, y^m), V^f(p^0, p^2, p^4, y^f)), \forall i,$$

and

$$\hat{\lambda} = \hat{\lambda}(\alpha, \gamma(\alpha), V^m(p^0, p^1, p^3, y^m), V^f(p^0, p^2, p^4, y^f)).$$

This gives us:

$$\begin{bmatrix} N_{xx} & -p \\ (5 \times 5) & (5 \times 1) \end{bmatrix} \begin{bmatrix} X_p & X_y & X_V & X_\gamma \\ (5 \times 5) & (5 \times 2) & (5 \times 2) & (5 \times 1) \end{bmatrix} + \begin{bmatrix} -\lambda I + N_{xp} & N_{xy} & N_{xV} & N_{x\gamma} \\ (5 \times 5) & (5 \times 5) & (5 \times 2) & (5 \times 2) & (5 \times 1) \\ -q' & i' & 0 & 0 \\ (1 \times 5) & (1 \times 2) & (1 \times 2) & (1 \times 1) \end{bmatrix} = \begin{matrix} 0 \\ (6 \times 10) \end{matrix}. \tag{2.25}$$

To understand the members of (2.25) define U_x , as before, as the (5×2) array of the

gradients of each person's utility function,

$$U_x \equiv \begin{bmatrix} \frac{\partial U^f}{\partial x^i} & \frac{\partial U^m}{\partial x^i} \end{bmatrix}_{\substack{(5 \times 2) \\ (5 \times 1) \quad (5 \times 1)}};$$

B_1 and B_2 as
 $(2 \times 2) \quad (2 \times 1)$

$$B_1 \equiv \begin{bmatrix} \gamma(\gamma - 1) \frac{g^m}{g^f} \frac{1}{g^f} (g^f)^\gamma (-1), & \gamma \frac{1}{g^f} (g^f)^\gamma (-1) \\ \gamma \frac{1}{g^f} (g^f)^\gamma (-1), & 0 \end{bmatrix}_{(2 \times 2)}$$

and

$$B_2 \equiv \begin{bmatrix} \frac{g^m}{g^f} (g^f)^\gamma (1 + \gamma \ln g^f) \\ (g^f)^\gamma \ln g^f \end{bmatrix}_{(2 \times 1)},$$

respectively, where $g^f = U^f(x^f) - V^f$ and $g^m = U^m(x^m) - V^m$ represent the respective utility gains to f and m in the solution to GN; V_p as the (2×5) matrix containing the effects of price changes on the disagreement utilities,

$$V_p \equiv \begin{bmatrix} \frac{\partial V^f}{\partial p^k} \\ \frac{\partial V^m}{\partial p^k} \end{bmatrix}_{(2 \times 5)} = \begin{bmatrix} \frac{\partial V^f}{\partial p^0} & 0 & \frac{\partial V^f}{\partial p^2} & 0 & \frac{\partial V^f}{\partial p^4} \\ \frac{\partial V^m}{\partial p^0} & \frac{\partial V^m}{\partial p^1} & 0 & \frac{\partial V^m}{\partial p^3} & 0 \end{bmatrix};$$

γ_p as a (1×5) vector of the partial derivatives of γ with respect to the prices,
 (1×5)

$$\gamma_p \equiv \left[\frac{\partial \gamma}{\partial p^k} \right]_{(1 \times 5)};$$

V_y as the (2×2) matrix containing the effects of nonwage income changes on the disagreement utilities,

$$V_y \equiv \begin{bmatrix} \frac{\partial V^f}{\partial y^m} & \frac{\partial V^f}{\partial y^f} \\ \frac{\partial V^m}{\partial y^m} & \frac{\partial V^m}{\partial y^f} \end{bmatrix}_{(2 \times 2)} = \begin{bmatrix} 0 & \frac{\partial V^f}{\partial y^f} \\ \frac{\partial V^m}{\partial y^m} & 0 \end{bmatrix};$$

and γ_y as a (1×2) vector of the partial derivatives of γ with respect to the nonwage incomes,

$$\gamma_y \equiv \begin{bmatrix} \frac{\partial \gamma}{\partial y^m} & \frac{\partial \gamma}{\partial y^f} \end{bmatrix}.$$

Then we can write the members of (2.25) as N_{xx} , the (5×5) Hessian of the Nash product function $N(x'; \alpha)$,

$$N_{xx} \equiv \begin{bmatrix} \frac{\partial^2 N}{\partial x^i \partial x^j} \end{bmatrix};$$

N_{xp} , the (5×5) matrix of second partial derivatives of $N(x'; \alpha)$ with respect to the prices,

$$N_{xp} \equiv \begin{bmatrix} \frac{\partial^2 N}{\partial x^i \partial p^k} \end{bmatrix} = U_x \begin{bmatrix} B_1 & V_p \\ (2 \times 2) & (2 \times 5) \end{bmatrix} + \begin{bmatrix} B_2 & \gamma_p \\ (2 \times 1) & (1 \times 5) \end{bmatrix};$$

N_{xy} , the (5×2) matrix of second partial derivatives of $N(x'; \alpha)$ with respect to the nonwage incomes,

$$N_{xy} \equiv \begin{bmatrix} \frac{\partial^2 N}{\partial x^i \partial y^m} : \frac{\partial^2 N}{\partial x^i \partial y^f} \end{bmatrix} = U_x \begin{bmatrix} B_1 & V_y \\ (2 \times 2) & (2 \times 2) \end{bmatrix} + \begin{bmatrix} B_2 & \gamma_y \\ (2 \times 1) & (1 \times 2) \end{bmatrix};$$

N_{xV} , the (5×2) matrix of second partial derivatives of $N(x'; \alpha)$ with respect to the disagreement utilities,

$$N_{xV} \equiv \begin{bmatrix} \frac{\partial^2 N}{\partial x^i \partial V^f} : \frac{\partial^2 N}{\partial x^i \partial V^m} \end{bmatrix} = U_x \begin{bmatrix} B_1 \\ (5 \times 2) & (2 \times 2) \end{bmatrix};$$

and $N_{x\gamma}$, the (5×1) vector of second partial derivatives of $N(x'; \alpha)$ with respect to γ ,

$$N_{x\gamma} \equiv \begin{bmatrix} \frac{\partial^2 N}{\partial x^i \partial \gamma} \end{bmatrix} = U_x \begin{bmatrix} B_2 \\ (5 \times 2) & (2 \times 1) \end{bmatrix};$$

The p matrices are given by $p = [p^i]$ and $p' = [p^j]$, while this time we have $q' = [\hat{q}^k]$; i' is a (1×2) vector of ones; λI is the (5×5) identity matrix multiplied by the

Lagrange multiplier $\hat{\lambda}$; X_p is the (5×5) matrix of uncompensated price effects,

$$X_p \equiv \left[\frac{\partial \hat{x}^j}{\partial p^k} \right];$$

X_y is the (5×2) matrix of the marginal impacts of nonwage incomes on demands,

$$X_y \equiv \left[\frac{\partial \hat{x}^j}{\partial y^m} ; \frac{\partial \hat{x}^j}{\partial y^f} \right];$$

X_V , the (5×2) matrix of the effects of changes in the disagreement utilities on demands,

$$X_V \equiv \left[\frac{\partial \hat{x}^j}{\partial V^f} \quad \frac{\partial \hat{x}^j}{\partial V^m} \right];$$

and X_γ , the (5×1) vector of the effect of changes in γ on demands,

$$X_\gamma \equiv \left[\frac{\partial \hat{x}^j}{\partial \gamma} \right];$$

Finally we have λ_p , the (1×5) matrix of the effects of price changes on the Lagrange multiplier,

$$\lambda_p \equiv \left[\frac{\partial \hat{\lambda}}{\partial p^k} \right];$$

λ_y , the (1×2) matrix of the effects of changes in nonwage incomes on the Lagrange multiplier,

$$\lambda_y \equiv \left[\frac{\partial \hat{\lambda}}{\partial y^m} ; \frac{\partial \hat{\lambda}}{\partial y^f} \right];$$

λ_V , the (1×2) matrix of the effects of changes in the disagreement utilities on the Lagrange multiplier,

$$\lambda_V \equiv \left[\frac{\partial \hat{\lambda}}{\partial V^m} \quad \frac{\partial \hat{\lambda}}{\partial V^f} \right];$$

and λ_{γ} , the effect of changes in γ on the Lagrange multiplier,
 (1×1)

$$\lambda_{\gamma} \equiv \left[\frac{\partial \hat{\lambda}}{\partial \gamma} \right].$$

Also note that $\underset{(1 \times 1)}{0}$ is just zero, while $\underset{(1 \times 2)}{0}$ and $\underset{(6 \times 10)}{0}$ are, respectively, the (1×2) vector of zeros and the (6×10) matrix of zeros.

Note that we can express the equations in (2.25) equivalently as

$$\begin{aligned} & \begin{bmatrix} N_{xx} & -p \\ (5 \times 5) & (5 \times 1) \end{bmatrix} \begin{bmatrix} X_p & X_y & X_V & X_{\gamma} \\ (5 \times 5) & (5 \times 2) & (5 \times 2) & (5 \times 1) \end{bmatrix} + \\ & \begin{bmatrix} -p' & 0 \\ (1 \times 5) & (1 \times 1) \end{bmatrix} \begin{bmatrix} \lambda_p & \lambda_y & \lambda_V & \lambda_{\gamma} \\ (1 \times 5) & (1 \times 2) & (1 \times 2) & (1 \times 1) \end{bmatrix} + \\ & \begin{bmatrix} -\lambda I + \underset{(5 \times 5)}{U_x} & \underset{(5 \times 2)}{U_x} \begin{bmatrix} B_1 & V_p \\ (2 \times 2) & (2 \times 5) \end{bmatrix} + \underset{(2 \times 1)}{B_2} \underset{(1 \times 5)}{\gamma_p} \\ (1 \times 5) & \underset{(1 \times 2)}{i'} \end{bmatrix} \begin{bmatrix} B_1 & V_y \\ (2 \times 2) & (2 \times 2) \end{bmatrix} + \underset{(2 \times 1)}{B_2} \underset{(1 \times 2)}{\gamma_y} \\ (5 \times 2) & \underset{(1 \times 2)}{0} \end{bmatrix} \begin{bmatrix} \underset{(5 \times 2)}{U_x} \underset{(2 \times 2)}{B_1} & \underset{(5 \times 2)}{U_x} \underset{(2 \times 1)}{B_2} \\ (5 \times 2) & (5 \times 2) \end{bmatrix} \\ & = \underset{(6 \times 10)}{0}. \end{aligned} \quad (2.26)$$

Throughout the ensuing discussion we alternate between (2.25) and (2.26) as facilitates a clearer exposition.

We now turn to an analysis of the properties of observed demand functions implied by the generalized Nash model. Given a parameter vector α , a corresponding value $\gamma(\alpha) \in [0, \infty)$ and a pair of disagreement utilities $V^m(p^0, p^1, p^3, y^m)$ and $V^f(p^0, p^2, p^4, y^f)$, the solution to the programme GN is a system of generalized Nash demand equations

$$\hat{x}^i = \hat{x}^i(\alpha, \gamma(\alpha), V^m(p^0, p^1, p^3, y^m), V^f(p^0, p^2, p^4, y^f)), \forall i,$$

as well as a solution for

$$\hat{\lambda} = \hat{\lambda}(\alpha, \gamma(\alpha), V^m(p^0, p^1, p^3, y^m), V^f(p^0, p^2, p^4, y^f)).$$

These generalized Nash demands describe the response of household demands to variations in prices, nonwage incomes, the value of the Nash relative utility γ , and the disagreement utilities. However, following our earlier argument, note that we do not observe variations in the value of γ nor in the disagreement utilities, nor do we observe the response of household demands to such variation. Instead we observe some demand system $x^i(\alpha), \forall i$, defined by

$$x^i(\alpha) = \hat{x}^i(\alpha, \gamma(\alpha), V^m(p^0, p^1, p^3, y^m), V^f(p^0, p^2, p^4, y^f)).$$

We also observe the response of these demands to variations in the prices and nonwage incomes, for some unobservable distribution of bargaining power, $\gamma(\alpha)$, and pair of disagreement utilities, $V^m(p^0, p^1, p^3, y^m)$ and $V^f(p^0, p^2, p^4, y^f)$. We now derive the structural properties, implied by the generalized Nash framework, of such observable demands.

Since the Hessian matrix, N_{xx} , in (2.26) is just a matrix of second partial derivatives of a continuously differentiable and quasiconcave objective function, the partitioned inverse of the bordered Nash Hessian can be expressed as $\begin{bmatrix} a_n & g_n \\ g'_n & k_n \end{bmatrix}$, where the standard proof of the symmetry and negative semi definiteness of a_n applies. We can therefore re-arrange the matrix equations (2.26) to obtain

$$\begin{bmatrix} X_p & X_y & X_V & X_\gamma \\ (5 \times 5) & (5 \times 2) & (5 \times 2) & (5 \times 1) \\ \lambda_p & \lambda_y & \lambda_V & \lambda_\gamma \\ (1 \times 5) & (1 \times 2) & (1 \times 2) & (1 \times 1) \end{bmatrix} = - \begin{bmatrix} a_n & g_n \\ g'_n & k_n \\ (5 \times 5) & (5 \times 1) \\ (1 \times 5) & (1 \times 1) \end{bmatrix} \times$$

$$\begin{bmatrix} -\lambda I + \underset{(5 \times 5)}{U_x} \underset{(5 \times 2)}{\left[\underset{(2 \times 2)(2 \times 5)}{B_1 V_p} + \underset{(2 \times 1)(1 \times 5)}{B_2 \gamma_p} \right]} & \underset{(5 \times 2)}{U_x} \underset{(2 \times 2)(2 \times 2)}{\left[\underset{(2 \times 2)(2 \times 2)}{B_1 V_y} + \underset{(2 \times 1)(1 \times 2)}{B_2 \gamma_y} \right]} & \underset{(5 \times 2)(2 \times 2)}{U_x B_1} & \underset{(5 \times 2)(2 \times 1)}{U_x B_2} \\ -q'_{(1 \times 5)} & i'_{(1 \times 2)} & 0_{(1 \times 2)} & 0_{(1 \times 1)} \end{bmatrix}. \quad (2.27)$$

Let $\underset{(2 \times 3)}{B} = \begin{bmatrix} \underset{(2 \times 2)}{B_1} : \underset{(2 \times 1)}{B_2} \end{bmatrix}$. From (2.27) we have

$$\underset{(5 \times 2)}{X_y} = - \underset{(5 \times 5)(5 \times 2)(2 \times 3)}{a_n U_x B} \begin{bmatrix} \underset{(2 \times 2)}{V_y} \\ \underset{(1 \times 2)}{\gamma_y} \end{bmatrix} - \underset{(5 \times 1)(1 \times 2)}{g_n i'},$$

and post-multiplying this expression by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ implies

$$\underset{(5 \times 1)}{X_{y^m}} - \underset{(5 \times 1)}{X_{y^f}} = \begin{bmatrix} \underset{(5 \times 2)(2 \times 2)}{X_V V_y} + \underset{(5 \times 1)(1 \times 2)}{X_\gamma \gamma_y} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \quad (2.28)$$

Condition (2.28) tells us that as long as

$$\frac{\partial \hat{x}^j}{\partial V^f} \frac{\partial V^f}{\partial y^f} \neq \frac{\partial \hat{x}^j}{\partial V^m} \frac{\partial V^m}{\partial y^m}$$

and

$$\frac{\partial \gamma}{\partial y^m} \neq \frac{\partial \gamma}{\partial y^f},$$

then

$$\underset{(5 \times 1)}{X_{y^m}} - \underset{(5 \times 1)}{X_{y^f}} \neq 0.$$

In this case there is no income-pooling under the generalized Nash model and individual nonwage incomes enter the Nash demand functions separately. However comparing (2.28) with (2.21) demonstrates that the magnitude of the difference between $\underset{(5 \times 1)}{X_{y^m}}$ and $\underset{(5 \times 1)}{X_{y^f}}$ will,

in general, differ between the generalized Nash and the collective models.

$$\text{To solve for } g_n \text{ , post-multiply } X_y = - \begin{matrix} a_n & U_x & B \\ (5 \times 5) & (5 \times 2) & (2 \times 3) \end{matrix} \begin{bmatrix} V_y \\ (2 \times 2) \\ \gamma_y \\ (1 \times 2) \end{bmatrix} - \begin{matrix} g_n & i' \\ (5 \times 1) & (1 \times 2) \end{matrix} \text{ by } \begin{matrix} i \\ (2 \times 1) \end{matrix}$$

and re-arrange to get

$$g_n = -\frac{1}{2} \begin{matrix} X_y \\ (5 \times 2) \end{matrix} + \begin{matrix} a_n & U_x & B \\ (5 \times 5) & (5 \times 2) & (2 \times 3) \end{matrix} \begin{bmatrix} V_y \\ (2 \times 2) \\ \gamma_y \\ (1 \times 2) \end{bmatrix} \begin{matrix} i \\ (2 \times 1) \end{matrix}.$$

From (2.27) we also have

$$X_p = - \begin{matrix} a_n \\ (5 \times 5) \end{matrix} \begin{matrix} -\lambda I + U_x & B \\ (5 \times 5) & (5 \times 2)(2 \times 3) \end{matrix} \begin{bmatrix} V_p \\ (2 \times 5) \\ \gamma_p \\ (1 \times 5) \end{bmatrix} + \begin{matrix} g_n & q' \\ (5 \times 1) & (1 \times 5) \end{matrix}.$$

Substituting for g_n in this expression and re-arranging implies

$$\begin{aligned} & X_p + \frac{1}{2} X_y \begin{matrix} i & q' \\ (2 \times 1) & (1 \times 5) \end{matrix} \\ &= \begin{matrix} a_n & \lambda I \\ (5 \times 5) & (5 \times 5) \end{matrix} + \begin{matrix} X_\gamma \\ (5 \times 1) \end{matrix} \left\{ \begin{matrix} \gamma_p \\ (1 \times 5) \end{matrix} + \frac{1}{2} \begin{matrix} \gamma_y & i & q' \\ (1 \times 2) & (2 \times 1) & (1 \times 5) \end{matrix} \right\} + \left\{ \begin{matrix} X_V & V_p \\ (5 \times 2) & (2 \times 5) \end{matrix} + \frac{1}{2} \begin{matrix} X_V & V_y & i & q' \\ (5 \times 2) & (2 \times 2) & (2 \times 1) & (1 \times 5) \end{matrix} \right\}. \end{aligned} \quad (2.29)$$

Condition (2.29) has exactly the same interpretation as condition (2.22), the only difference being the presence of the extra term, $X_V V_p + \frac{1}{2} X_V V_y \begin{matrix} i & q' \\ (2 \times 1) & (1 \times 5) \end{matrix}$, on the right hand side of (2.29). This term represents the effect of a price change, compensated by an increase in income, on the disagreement utilities, and the consequent effect this has on household consumption and leisure demands. Note also that the matrix $\begin{matrix} a_n & \lambda I \\ (5 \times 5) & (5 \times 5) \end{matrix}$ is the symmetric and negative semi definite matrix of substitution effects, with household utility, the relative utility weight γ and the disagreement utilities held constant.

We can now summarize the main restrictions on any system of household demands implied by the generalized Nash model as follows:

1. The Slutsky matrix $X_p + \frac{1}{2} X_y \begin{smallmatrix} i \\ q' \end{smallmatrix}$ need not be symmetric and negative semi definite.

2. The Slutsky matrix is decomposed according to

$$\begin{aligned} & X_p + \frac{1}{2} X_y \begin{smallmatrix} i \\ q' \end{smallmatrix} \\ & \begin{smallmatrix} (5 \times 5) & (5 \times 2)(2 \times 1)(1 \times 5) \end{smallmatrix} \\ & = \begin{smallmatrix} a_n & \lambda I \\ (5 \times 5) & (5 \times 5) \end{smallmatrix} + \left\{ \begin{smallmatrix} X_\gamma & \gamma_p \\ (5 \times 1) & (1 \times 5) \end{smallmatrix} + \frac{1}{2} \begin{smallmatrix} X_\gamma & \gamma_y & i & q' \\ (5 \times 1) & (1 \times 2) & (2 \times 1) & (1 \times 5) \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} X_V & V_p \\ (5 \times 2) & (2 \times 5) \end{smallmatrix} + \frac{1}{2} \begin{smallmatrix} X_V & V_y & i & q' \\ (5 \times 2) & (2 \times 2) & (2 \times 1) & (1 \times 5) \end{smallmatrix} \right\}. \end{aligned} \quad (2.29)$$

3. As long as $\frac{\partial \hat{x}^j}{\partial V^f} \frac{\partial V^f}{\partial y^f} \neq \frac{\partial \hat{x}^j}{\partial V^m} \frac{\partial V^m}{\partial y^m}$ and $\frac{\partial \gamma}{\partial y^m} \neq \frac{\partial \gamma}{\partial y^f}$, generalized Nash bargaining implies the absence of income pooling within the household, i.e.

$$\begin{smallmatrix} X_{y_m} & - & X_{y_f} \\ (5 \times 1) & & (5 \times 1) \end{smallmatrix} = \begin{bmatrix} \begin{smallmatrix} X_V & V_y \\ (5 \times 2) & (2 \times 2) \end{smallmatrix} + \begin{smallmatrix} X_\gamma & \gamma_y \\ (5 \times 1) & (1 \times 2) \end{smallmatrix} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \neq 0. \quad (2.28)$$

This completes our analysis of the restrictions on household demand systems implied by the generalized Nash model of household decision making.

2.4.3 The Relationship between Alternative Models of Household Decision Making

Before we discuss the relationship between the collective and the generalized Nash models, we recap some standard results from the traditional and the symmetric Nash models of household decision making. McElroy (1990) derived the fundamental comparative static equations for a traditional model of the household¹⁹. Her equations were augmented to take account of the effect of ‘extrahousehold environmental parameters’, or EEPs, on

¹⁹See McElroy (1990), p. 569, equation (12).

the traditional household demand system. EEPs are parameters, apart from prices and nonwage incomes, that may also affect the household consumption and leisure demands. However disregarding the effects of changes in these EEPs it is easy to verify that the traditional comparative static equations are given by

$$\begin{bmatrix} \tilde{U}_{xx} & -p \\ (5 \times 5) & (5 \times 1) \end{bmatrix} \begin{bmatrix} X_p & X_y \\ (5 \times 5) & (5 \times 2) \end{bmatrix} + \begin{bmatrix} -\lambda I & 0 \\ (5 \times 5) & (5 \times 2) \end{bmatrix} = \begin{bmatrix} 0 \\ (6 \times 7) \end{bmatrix}, \quad (2.30)$$

where \tilde{U}_{xx} represents the traditional Hessian matrix containing the second partial derivatives of the single, neo-classical, utility function that represents household preferences. The remaining members of (2.30) are analogous to those defined earlier. From (2.30) it is easy to see that under the traditional model:

1. The Slutsky matrix is decomposed according to

$$\begin{bmatrix} X_p \\ (5 \times 5) \end{bmatrix} + \begin{bmatrix} X_y & q' \\ (5 \times 1) & (1 \times 5) \end{bmatrix} = \begin{bmatrix} \tilde{a} & \lambda I \\ (5 \times 5) & (5 \times 5) \end{bmatrix}, \quad (2.31)$$

where $\begin{bmatrix} \tilde{a} & \lambda I \\ (5 \times 5) & (5 \times 5) \end{bmatrix}$ is a symmetric and negative semi definite matrix.

2. There is income-pooling within the household, i.e.

$$\begin{bmatrix} X_{y_m} \\ (5 \times 1) \end{bmatrix} - \begin{bmatrix} X_{y_f} \\ (5 \times 1) \end{bmatrix} = 0. \quad (2.32)$$

McElroy (1990) also obtained the fundamental equations for a symmetric Nash model of the household²⁰ which we reproduce here, again ignoring the effects of changes in the EEPs on the symmetric Nash household demand system.

$$\begin{bmatrix} N_{xx}^S & -p \\ (5 \times 5) & (5 \times 1) \end{bmatrix} \begin{bmatrix} X_p & X_y & X_V \\ (5 \times 5) & (5 \times 2) & (5 \times 2) \end{bmatrix} + \begin{bmatrix} -p' & 0 \\ (1 \times 5) & (1 \times 1) \end{bmatrix} \begin{bmatrix} \lambda_p & \lambda_y & \lambda_V \\ (1 \times 5) & (1 \times 2) & (1 \times 2) \end{bmatrix}$$

²⁰See McElroy (1990), p. 568, equation (11).

$$\begin{aligned}
& + \begin{bmatrix} -\lambda I + \begin{smallmatrix} U_x & B_1^S & V_p \\ (5 \times 5) & (5 \times 2)(2 \times 2)(2 \times 5) \end{smallmatrix} & \begin{smallmatrix} U_x & B_1^S & V_y \\ (5 \times 2)(2 \times 2)(2 \times 2) \end{smallmatrix} & \begin{smallmatrix} U_x & B_1^S \\ (5 \times 2)(2 \times 2) \end{smallmatrix} \\ -q' & & 0 \\ (1 \times 5) & (1 \times 2) & (1 \times 2) \end{bmatrix} \\
& = \begin{smallmatrix} 0 \\ (6 \times 10) \end{smallmatrix},
\end{aligned} \tag{2.33}$$

where N_{xx}^S represents the symmetric Nash Hessian and $B_1^S \equiv - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The remaining members of (2.33) are analogous to those defined earlier. Under the symmetric Nash model we have:

1. The Slutsky matrix $X_p + \frac{1}{2} X_y \begin{smallmatrix} i & q' \\ (5 \times 2)(2 \times 1)(1 \times 5) \end{smallmatrix}$ need not be symmetric and negative semi definite.
2. The Slutsky matrix is decomposed according to

$$X_p + \frac{1}{2} X_y \begin{smallmatrix} i & q' \\ (5 \times 2)(2 \times 1)(1 \times 5) \end{smallmatrix} = a_n^S \lambda I + \left\{ X_V \begin{smallmatrix} V_p \\ (5 \times 2)(2 \times 5) \end{smallmatrix} + \frac{1}{2} X_V \begin{smallmatrix} V_y & i & q' \\ (5 \times 2)(2 \times 2)(2 \times 1)(1 \times 5) \end{smallmatrix} \right\}, \tag{2.34}$$

where $a_n^S \lambda I$ is a symmetric and negative semi definite matrix.

3. Symmetric Nash bargaining implies the absence of income-pooling within the household, i.e.

$$X_{y_m} - X_{y_f} = X_V \begin{smallmatrix} V_y \\ (5 \times 2)(2 \times 2) \end{smallmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \neq 0. \tag{2.35}$$

We are now in a position to examine the relationship between all four models discussed in this chapter, namely the traditional, the symmetric Nash, the generalized Nash and the collective models of household decision making. Suppose, in the generalized Nash model, we set $\gamma_{p^k} = \gamma_{y^h} = 0, \forall k, h$. This restricts the generalized Nash utility weight to being a constant with respect to changes in the exogenous parameters and effectively restricts the household to some arbitrary point on the Pareto frontier. Furthermore if we set $\gamma(\alpha) = 1$, each person's preferences receive equal weight in the overall household objective, thus equalizing the utility gains across household members in the solution

to the model. Under these restrictions of the generalized Nash model we obtain the symmetric Nash model of MHM. Comparing (2.28) with (2.35) and (2.29) with (2.34) we see that under a constant relative utility weight, the generalized Nash comparative statics collapse to those of the symmetric Nash framework. In other words under these restrictions of the generalized Nash model, both the generalized Nash and the symmetric Nash models imply the same structural properties for household demand systems.

MHM demonstrated that restriction of the symmetric Nash model so that $\frac{\partial V^f}{\partial p^k} = \frac{\partial V^m}{\partial p^k} = \frac{\partial V^f}{\partial y^f} = \frac{\partial V^m}{\partial y^m} = 0$, $\forall k$, (i.e. constant disagreement utilities) implies that the comparative statics of the symmetric Nash model collapse to those of the traditional model and both models imply the same restrictions on household demand systems. Compare (2.31) with (2.34) and (2.32) with (2.35) to see that this is indeed the case.

Now suppose, in the generalized Nash model, we set $\frac{\partial V^f}{\partial p^k} = \frac{\partial V^m}{\partial p^k} = \frac{\partial V^f}{\partial y^f} = \frac{\partial V^m}{\partial y^m} = 0$, $\forall k$. In this case the generalized Nash comparative statics collapse to the comparative statics of the collective model. Compare (2.28) with (2.21) and (2.29) with (2.22). Finally suppose, in the collective model, we set $\frac{\partial \mu}{\partial p^k} = \frac{\partial \mu}{\partial y^h} = 0$, $\forall k, h$. This restricts the relative utility weight of the collective model, $\mu(\alpha)$, to being a constant with respect to changes in the exogenous parameters and therefore restricts the household to being at a particular point on the Pareto frontier. Comparing (2.21) with (2.32) and (2.22) with (2.31) we see that when $\mu(\alpha)$ is constant, the comparative statics of the collective model collapse to those of the traditional framework.

We can therefore combine the results of MHM with our own results to establish the following relationship between all four models: the comparative statics of the traditional model are nested within those of the symmetric Nash model, which in turn are nested within those of the generalized Nash model. Also the comparative statics of the traditional model are nested within those of the collective model, which in turn are nested within those of the generalized Nash.

Our analysis suggests that, of the four models discussed in this chapter, the generalized Nash model offers the most general analysis of the intrahousehold decision process

and is the least restrictive of the household demand systems. The generalized Nash model preserves the main characteristic of the collective model: it uses knowledge of the individual utilities and of the household budget constraint to generate a continuum of Pareto efficient outcomes and it does not restrict the household *a priori* to any particular point on the Pareto frontier. In this respect it eliminates one of the main restrictions of the symmetric Nash model. In addition, the generalized Nash model generalizes the household decision problem even further by effectively eliminating one of the main restrictions of the collective approach: the generalized Nash model allows for the *possibility* of disagreement among household members and for the opportunity cost of a person's household membership to influence any cooperative outcome. In this respect the generalized Nash model offers a way towards an even more general characterization of the household decision process, by allowing a 'theory of household dissolution' to be incorporated and to have some influence over household decisions. This possibility is not allowed for within the collective framework as currently specified.²¹

We may interpret the collective approach to household decisions as being any approach that generates a continuum of outcomes along the Pareto frontier but excludes the possibility of household dissolution and the role that the opportunity cost of household membership may play in influencing the household's final location on the Pareto frontier. On the other hand we may interpret the generalized Nash approach, presented here, as not only generating the Pareto frontier but also as allowing for the possibility of disagreement and the effect this may have on the household decision process. The collective approach is concerned solely with cooperative behaviour within the household,

²¹ However it is not difficult to extend the collective model to incorporate a pair of disagreement utilities. In this case the household decision problem, stated in C'2, may look something like

$$\max_{x^i, \forall i} \{ (U^m(x^m) - V^m) + \mu(\alpha) (U^f(x^f) - V^f) / y^m + y^f - p'q \geq 0 \},$$

where V^m and V^f are the disagreement utilities of m and f , respectively, and are functions of the relevant exogenous parameters. In terms of its implied properties of observable household demands, a collective model thus defined would be exactly equivalent to the generalized Nash model presented in GN.

whereas the generalized Nash approach incorporates both the full range of possible cooperative outcomes as well as a possible noncooperative outcome. It also allows for a meaningful specification of the minimum cooperative payoffs in terms of the *individual rationality* of each player, rather than on pure feasibility criteria.

We raise one final point with regard to the interpretation of our results. Throughout we have argued that a person's disagreement utility can be interpreted as their opportunity cost of household membership, i.e. as the utility/satisfaction they could receive under some alternative household arrangement. Also we have shown that in the generalized Nash model, these disagreement utilities appear as arguments in the household demand functions and therefore influence the household decision process. However, first note that there may be quite a large variation in the value of the disagreement utilities with no corresponding effect on the intrahousehold allocation. This idea becomes clearer if we think of the household problem geometrically, in utility space. Second, there is no reason why the generalized Nash utility weight, γ , cannot depend directly on each individual's opportunity cost of household membership. Note, however, that if we allow γ to depend on these opportunity costs and we dispense with the disagreement utilities, then we are back to a purely cooperative approach with no role for disagreement and household dissolution. This would still be a more restrictive approach than the generalized Nash approach advocated here.

2.5 Discussion and Areas for Further Work

The analysis of this chapter suggests several interesting avenues for future research. First among these must be to develop tests of the collective framework presented in CH *vis-a-vis* a framework, such as the generalized Nash model presented here, that explicitly allows for disagreement and opportunity costs to influence the final intrahousehold outcome. In this chapter we have derived the Slutsky equations associated with the collective and the generalized Nash models and have demonstrated that the *decomposition* of the

compensated price effects will be different in both cases. It remains to translate this difference into empirical tests that allow us to distinguish between the collective approach and an approach such as the generalized Nash. In particular we want to establish whether, and to what extent, outside options influence decisions within the household. We also want to understand more about the nature and the content of these outside options, what role there may be for state-dependent preferences, and the influence of social customs and of the legal system. Finally we also want to establish which outside factors have a strong significance and which ones less so.

Second, is the broader question of the formation and dissolution of households. In this chapter we see that whereas the Nash approach explicitly addresses the question of the household's failure to reach a cooperative agreement, albeit in a very simplified fashion, the collective approach ignores this question altogether. Our 'model' of household dissolution followed the literature at large in assuming (a) a very simple division of the total household nonwage income (i.e. each family member takes precisely what they contributed in the first place), and (b) that individual utility is independent of the 'marital state'. However such an approach entails particular and strong assumptions about the cultural/legal framework governing separation, about the nature of divorce and of family dissolution in general, about the enforcement of laws, and about individual preferences over their marital state.²² The development of a proper treatment of the formation and dissolution of households, from which we can derive an appropriate set of disagreement utilities, presents another exciting area for future research to improve our understanding of household decision processes.

APPENDIX

Proof of Lemma 1. (i) U closed follows from X being closed. (ii) U is bounded as long as all elements in X yield a finite utility to both members. (iii) Strict convexity of U

²² It is also implicit in the Nash analysis that family members are individually rational and can choose not to accept the cooperative solution if that is in their interest. Therefore the Nash solution must offer a utility payoff that is no less than some privately optimal level.

follows from the strict concavity of the individual utility functions and the convexity of X . Let the vectors x , x^m and x^f be as defined earlier. Let \hat{x} , \hat{x}^m and \hat{x}^f be corresponding vectors in which x^i is replaced by \hat{x}^i , $\forall i$. Then strict concavity of U^h ($h = m, f$) implies that for any pair of consumption allocations $x, \hat{x} \in X$ and for all $a \in [0, 1]$, we have

$$U^m(ax^m + (1-a)\hat{x}^m) > aU^m(x^m) + (1-a)U^m(\hat{x}^m)$$

and

$$U^f(ax^f + (1-a)\hat{x}^f) > aU^f(x^f) + (1-a)U^f(\hat{x}^f).$$

Consider any two feasible utility allocations

$$(U^m(x^m), U^f(x^f)) \in \mathbf{U}$$

and

$$(U^m(\hat{x}^m), U^f(\hat{x}^f)) \in \mathbf{U}.$$

Then the consumption allocations x and \hat{x} must both belong to X and we must have

$$y^m + y^f + (p^3 + p^4)T \geq p'x$$

and

$$y^m + y^f + (p^3 + p^4)T \geq p'\hat{x}.$$

Now consider the average of the feasible consumption allocations x and \hat{x} , given by

$$\begin{aligned} & ax + (1-a)\hat{x} \\ = & (ax^0 + (1-a)\hat{x}^0, ax^1 + (1-a)\hat{x}^1, ax^2 + (1-a)\hat{x}^2, ax^3 + (1-a)\hat{x}^3, ax^4 + (1-a)\hat{x}^4). \end{aligned}$$

Since the set X is convex we must have

$$\begin{aligned}
& p' (ax + (1 - a) \hat{x}) \\
&= \sum_i p^i (ax^i + (1 - a) \hat{x}^i) \\
&= a \sum_i p^i x^i + (1 - a) \sum_i p^i \hat{x}^i \\
&= ap'x + (1 - a) p' \hat{x} \\
&\leq y^m + y^f + (p^3 + p^4) T,
\end{aligned}$$

and therefore $ax + (1 - a) \hat{x} \in X$. But then it follows that

$$(U^m(ax^m + (1 - a) \hat{x}^m), U^f(ax^f + (1 - a) \hat{x}^f)) \in \mathbf{U}.$$

From the strict concavity of U^h ($h = m, f$) we have

$$U^m(ax^m + (1 - a) \hat{x}^m) > aU^m(x^m) + (1 - a)U^m(\hat{x}^m)$$

and

$$U^f(ax^f + (1 - a) \hat{x}^f) > aU^f(x^f) + (1 - a)U^f(\hat{x}^f).$$

Therefore we must have

$$\begin{aligned}
& (aU^m(x^m) + (1 - a)U^m(\hat{x}^m), aU^f(x^f) + (1 - a)U^f(\hat{x}^f)) \\
&= a(U^m(x^m), U^f(x^f)) + (1 - a)(U^m(\hat{x}^m), U^f(\hat{x}^f)) \\
&\in \mathbf{U}, \\
&\forall a \in [0, 1].
\end{aligned}$$

We have just shown that, for any pair of feasible utility allocations

$$(U^m(x^m), U^f(x^f)) \in \mathbf{U}$$

and

$$(U^m(\hat{x}^m), U^f(\hat{x}^f)) \in \mathbf{U},$$

and for any $a \in [0, 1]$, we must have

$$[a(U^m(x^m), U^f(x^f)) + (1-a)(U^m(\hat{x}^m), U^f(\hat{x}^f))] \in \mathbf{U}.$$

■

Proof of Lemma 2. Under the solutions to (2.3) and (2.4) note that the individuals' budget constraint must be binding since utility maximization requires all wage and non-wage income to be spent. Suppose m and f now form a joint household. Then m and f pool their resources to form the joint household full income constraint which is just the sum of their individual budget constraints. Since V_0^m and V_0^f were the private utility payoffs to m and f , respectively, we assume that neither household member will accept a cooperative outcome that offers less than their respective private payoff.

We now show that if, in the joint household, person f receives V_0^f , then person m must receive a payoff, \hat{V}^m , that is exactly equal to V_0^m . Since there are no shared goods in the joint household, \hat{V}^m must be defined by

$$\hat{V}^m = \max_{x^i, i=1, \dots, 4} \left\{ \begin{array}{l} U^m(0, x^1, x^3) / U^f(0, x^2, x^4) = V_0^f \text{ and} \\ y^m + y^f + (p^3 + p^4)T \geq p^1x^1 + p^2x^2 + p^3x^3 + p^4x^4. \end{array} \right\} \quad (2.36)$$

Suppose, in the solution to (2.36), (x^2, x^4) is such that $U^f(0, x^2, x^4) = V_0^f$ and $p^2x^2 + p^4x^4 = y^f + p^4T$. Then, from the strict concavity of $U^f(x^f)$ and the convexity of f 's private budget constraint, (x^2, x^4) must equal (x^{2*}, x^{4*}) , the solution to (2.4). In this case (x^1, x^3) , in the solution to (2.36), must also equal (x^{1*}, x^{3*}) , the solution to (2.3), and we must have $U^m(0, x^1, x^3) = V_0^m = \hat{V}^m$. Suppose, instead, in the solution to (2.36), person f receives some other allocation $(\hat{x}^2, \hat{x}^4) \neq (x^{2*}, x^{4*})$ such that $U^f(0, \hat{x}^2, \hat{x}^4) = V_0^f$. Then, again from the strict concavity of $U^f(x^f)$ and the convexity of f 's private budget constraint, it must be the case that (\hat{x}^2, \hat{x}^4) lies outside of f 's set of privately feasible

consumption allocations, i.e. we must have $p^2\hat{x}^2 + p^4\hat{x}^4 > y^f + p^4T$. However since the joint household budget constraint is binding in the solution to (2.36), then (\hat{x}^1, \hat{x}^3) , in the solution to this problem, must be such that $y^m + p^3T > (p^1\hat{x}^1 + p^3\hat{x}^3)$. But any (\hat{x}^1, \hat{x}^3) such that $y^m + p^3T > (p^1\hat{x}^1 + p^3\hat{x}^3)$ must lead to a lower utility payoff to m than $V_0^m = U^m(0, x^{1*}, x^{3*})$. This follows from the strict concavity of $U^m(x^m)$ and the convexity of m 's private budget constraint. It follows that $U^m(0, \hat{x}^1, \hat{x}^3) < V_0^m = U^m(0, x^{1*}, x^{3*})$. Intuitively, remember that m 's private budget constraint was binding under the solution to m 's privately optimal problem. This solution led to a utility payoff $V_0^m = U^m(0, x^{1*}, x^{3*})$ for m . Any tightening of m 's individual constraint (or, equivalently, any leaving of income unspent) must lead to a new allocation that reduces the utility payoff to m .

We have now shown that if, in the solution to (2.36), f receives a consumption allocation that yields V_0^f , then m must receive a consumption allocation that yields V_0^m or less. However, since m will not accept any utility payoff that is strictly less than V_0^m , then m will receive V_0^m exactly. It is therefore not feasible for m to receive strictly more than V_0^m if f receives V_0^f . Following a symmetric argument, we can also show that it is not feasible for f to receive strictly more than V_0^f if m receives V_0^m . We have now shown that whenever f receives V_0^f , m receives V_0^m exactly, and whenever m receives V_0^m , f receives V_0^f exactly. In both cases, m and f receive the consumption allocation (x^{1*}, x^{3*}) and (x^{2*}, x^{4*}) , respectively. In other words m and f receive the allocations that are solutions to (2.3) and (2.4), respectively.

Now suppose f receives a consumption allocation that yields strictly more than V_0^f . Then it is easy to see that this can only be achieved if m receives a consumption allocation that yields strictly less than V_0^m . By a symmetric argument, if m receives a consumption allocation that yields strictly more than V_0^m , then f must receive a consumption allocation that yields strictly less than V_0^f . Therefore in the absence of shared goods we have shown that the utility allocation (V_0^m, V_0^f) must lie on the Pareto frontier. When there are no shared goods there are therefore no utility payoffs to be obtained in a joint household,

over and above the utility payoffs that are privately achievable. ■

Proof of Lemma 3. We establish the strict inequalities. When there are shared goods then, within the joint household, we must have $x_m^0 = x_f^0 = x^0$. Hence if m and f form a joint household their joint decision problem can be represented by

$$\max_{x^i, i=0,1,\dots,4} \left\{ \begin{array}{l} U^m(x^0, x^1, x^3) / \\ U^f(x^0, x^2, x^4) \geq V^f \text{ and} \\ y^m + y^f + (p^3 + p^4)T \geq p'x \end{array} \right\}. \quad (2.37)$$

Let $(\hat{x}_m^0, \hat{x}^1, \hat{x}^3)$ and $(\hat{x}_f^0, \hat{x}^2, \hat{x}^4)$ represent the solutions to the private constrained maximization problems in (2.1) and (2.2), respectively, with V^m and V^f representing the corresponding utility payoffs. Then these solutions must satisfy

$$p^0 \hat{x}_m^0 + p^1 \hat{x}^1 + p^3 \hat{x}^3 = y^m + p^3 T \quad (2.38)$$

and

$$p^0 \hat{x}_f^0 + p^2 \hat{x}^2 + p^4 \hat{x}^4 = y^f + p^4 T, \quad (2.39)$$

and we must have $(V^m, V^f) \in \mathbf{U}$. Combining (2.38) and (2.39) implies

$$p^0 (\hat{x}_m^0 + \hat{x}_f^0) + p^1 \hat{x}^1 + p^2 \hat{x}^2 + p^3 \hat{x}^3 + p^4 \hat{x}^4 = y^m + y^f + p^3 T + p^4 T. \quad (2.40)$$

Let $x^{i*}, i = 0, \dots, 4$, represent the solution to the joint household problem (2.37). Then this solution must satisfy

$$p^0 x^{0*} + p^1 x^{1*} + p^2 x^{2*} + p^3 x^{3*} + p^4 x^{4*} = y^m + y^f + (p^3 + p^4) T. \quad (2.41)$$

Now consider the joint household allocation, \bar{x} , given by $\bar{x}^i = \hat{x}^i, i = 1, \dots, 4$, and $\bar{x}^0 = \hat{x}_m^0 + \hat{x}_f^0$. Such an allocation is clearly feasible under the joint household budget. However note that as long as $\hat{x}_h^0 > 0$ ($h = m, f$) then we must also have $\bar{x}^0 > \hat{x}_h^0$ ($h = m, f$).

Therefore there exists a feasible household allocation, the allocation \bar{x} , that provides each person with exactly the same quantities of $x^i, i = 1, \dots, 4$, consumed privately, and each person with strictly more of x^0 than was privately consumed. The utility payoffs to m and f from this feasible allocation, \bar{x} , are given by $\hat{U}^m = U^m(\hat{x}_m^0 + \hat{x}_f^0, \hat{x}^1, \hat{x}^3)$ and $\hat{U}^f = U^f(\hat{x}_m^0 + \hat{x}_f^0, \hat{x}^2, \hat{x}^4)$, respectively. Since individual utilities are strictly increasing in each of their arguments, then these utility payoffs must provide the household members with strictly greater utility than the utility they obtained privately. It follows that $(\hat{U}^m, \hat{U}^f) \in U$ and $\hat{U}^h > V^h$ ($h = m, f$). Therefore we have just shown that if there is a shared good and there are interior solutions for this shared good when individuals act privately, then there exists a feasible, non-empty set of utility allocations, under the joint household, that provides each person with strictly more than their privately optimal utilities. ■

Proof of Lemma 4. (i) X convex follows from the linear household budget constraint. (ii) That u^f belongs to the compact interval $[u_{\min}^f, u_{\max}^f]$ was demonstrated in section 2.3.1. (iii) That $F(\cdot, \alpha)$ is decreasing in u^f can be seen by applying the Envelope theorem to (2.5). The strict concavity of $F(\cdot, \alpha)$ in u^f follows from the strict convexity of U . Let \hat{u} represent some feasible utility payoff to person f , i.e. $\hat{u} \in (u_{\min}^f, u_{\max}^f)$, and let $\mu(\hat{u}, \alpha)$ be the value of the multiplier on f 's utility constraint in the solution to (2.5), where this problem is evaluated at $u^f = \hat{u}$. We can verify that $\mu(\hat{u}, \alpha) = \frac{p^i U_{x^j}^m}{p^j U_{x^i}^f} > 0$ ($i = 2, 4$ and $j = 1, 3$), where the right hand side of this expression is evaluated at the solution to (2.5). Now since $u_{\max}^f > u_{\min}^f$, there exists $u^f \in (u_{\min}^f, u_{\max}^f)$ such that $u^f \neq \hat{u}$. Since $F(\cdot, \alpha)$ is defined on (u_{\min}^f, u_{\max}^f) then $F(\cdot, \alpha)$ must be differentiable at \hat{u} . To see that this must be true note that as $u^f = \hat{u} + du$ varies,

$$\lim_{u^f \rightarrow \hat{u}} \frac{F(u^f) - F(\hat{u})}{u^f - \hat{u}} = \lim_{du \rightarrow 0} \frac{F(\hat{u} + du) - F(\hat{u})}{du}$$

exists and is given by

$$F'(\hat{u}, \alpha) = -\mu(\hat{u}) < 0.$$

Since $F(\cdot, \alpha)$ is differentiable at \hat{u} it must also be continuous at \hat{u} . But since $F(\cdot, \alpha)$ is monotonic in u^f then $F(\cdot, \alpha)$ must be continuously differentiable. (iv) This is obvious given the strict concavity of the individual utilities and the convexity of X . ■

Derivation of the Collective Comparative Static Equations. To obtain (2.19), totally differentiate each of the first order conditions in (2.17) and (2.18) with respect to each of the parameters in $(\alpha, \mu(\alpha))$. This gives us the following set of equations, evaluated at the solution $x^{i*}(\alpha, \mu(\alpha))$, $\forall i$, and $\lambda^*(\alpha, \mu(\alpha))$:

$$\begin{aligned} \sum_j \left\{ \left(\frac{\partial^2 U^m}{\partial x^i \partial x^j} + \mu(\alpha) \frac{\partial^2 U^f}{\partial x^i \partial x^j} \right) \frac{\partial x^j}{\partial p^k} \right\} - p^i \frac{\partial \lambda}{\partial p^k} + \frac{\partial U^f}{\partial x^i} \frac{\partial \mu}{\partial p^k} - z &= 0, \forall i, k, \\ \text{where } z &= \lambda, \forall i = k, \\ z &= 0, \forall i \neq k, \\ \sum_j \left\{ \left(\frac{\partial^2 U^m}{\partial x^i \partial x^j} + \mu(\alpha) \frac{\partial^2 U^f}{\partial x^i \partial x^j} \right) \frac{\partial x^j}{\partial y^h} \right\} - p^i \frac{\partial \lambda}{\partial y^h} + \frac{\partial U^f}{\partial x^i} \frac{\partial \mu}{\partial y^h} &= 0, \forall i, h, \\ \sum_j \left\{ \left(\frac{\partial^2 U^m}{\partial x^i \partial x^j} + \mu(\alpha) \frac{\partial^2 U^f}{\partial x^i \partial x^j} \right) \frac{\partial x^j}{\partial \mu} \right\} - p^i \frac{\partial \lambda}{\partial \mu} + \frac{\partial U^f}{\partial x^i} &= 0, \forall i, \\ - \sum_j p^j \frac{\partial x^j}{\partial p^k} + 0 \frac{\partial \lambda}{\partial p^k} - q^k &= 0, \forall k, \\ - \sum_j p^j \frac{\partial x^j}{\partial y^h} + 0 \frac{\partial \lambda}{\partial y^h} + 1 &= 0, \forall h, \\ - \sum_j p^j \frac{\partial x^j}{\partial \mu} + 0 \frac{\partial \lambda}{\partial \mu} &= 0. \end{aligned}$$

These equations can be arranged into the matrix equations (2.19).

Derivation of the Generalized Nash Comparative Static Equations. To obtain the comparative static equations for the generalized Nash model, i.e. equations (2.25) and (2.26), totally differentiate each of the first order conditions (2.23) and (2.24) with respect to each of the parameters in

$$(\alpha, \gamma(\alpha), V^m(p^0, p^1, p^3), V^f(p^0, p^2, p^4)),$$

evaluating the resulting equations at the solution to GN given by

$$\hat{x}^i = \hat{x}^i(\alpha, \gamma(\alpha), V^m(p^0, p^1, p^3), V^f(p^0, p^2, p^4)), \forall i$$

and

$$\hat{\lambda} = \hat{\lambda}(\alpha, \gamma(\alpha), V^m(p^0, p^1, p^3), V^f(p^0, p^2, p^4)).$$

This gives us

$$\begin{aligned} \sum_j \left\{ \frac{\partial^2 N}{\partial x^i \partial x^j} \frac{\partial x^j}{\partial p^k} \right\} - p^i \frac{\partial \lambda}{\partial p^k} + \frac{\partial^2 N}{\partial x^i \partial p^k} - z &= 0, \forall i, k, \\ \text{where } z &= \lambda, \forall i = k, \\ z &= 0, \forall i \neq k, \\ \sum_j \left\{ \frac{\partial^2 N}{\partial x^i \partial x^j} \frac{\partial x^j}{\partial y^h} \right\} - p^i \frac{\partial \lambda}{\partial y^h} + \frac{\partial^2 N}{\partial x^i \partial y^h} &= 0, \forall i, h, \\ \sum_j \left\{ \frac{\partial^2 N}{\partial x^i \partial x^j} \frac{\partial x^j}{\partial V^h} \right\} - p^i \frac{\partial \lambda}{\partial V^h} + \frac{\partial^2 N}{\partial x^i \partial V^h} &= 0, \forall i, h, \\ \sum_j \left\{ \frac{\partial^2 N}{\partial x^i \partial x^j} \frac{\partial x^j}{\partial \gamma} \right\} - p^i \frac{\partial \lambda}{\partial \gamma} + \frac{\partial^2 N}{\partial x^i \partial \gamma} &= 0, \forall i, \\ - \sum_j p^j \frac{\partial x^j}{\partial p^k} + 0 \frac{\partial \lambda}{\partial p^k} - q^k &= 0, \forall k, \\ - \sum_j p^j \frac{\partial x^j}{\partial y^h} + 0 \frac{\partial \lambda}{\partial y^h} + 1 &= 0, \forall h, \\ - \sum_j p^j \frac{\partial x^j}{\partial V^h} + 0 \frac{\partial \lambda}{\partial V^h} &= 0, \forall h, \\ - \sum_j p^j \frac{\partial x^j}{\partial \gamma} + 0 \frac{\partial \lambda}{\partial \gamma} &= 0. \end{aligned}$$

Note that the second partial derivatives of the Nash product function are given by

$$\frac{\partial^2 N}{\partial x^i \partial x^j} = (g^f)^\gamma \left(\frac{\partial^2 U^m}{\partial x^i \partial x^j} + \gamma \frac{g^m}{g^f} \frac{\partial^2 U^f}{\partial x^i \partial x^j} + (\gamma - 1) \gamma \frac{1}{g^f} \frac{g^m}{g^f} \frac{\partial U^f}{\partial x^j} \frac{\partial U^f}{\partial x^i} \right. \\ \left. + \gamma \frac{1}{g^f} \left(\frac{\partial U^m}{\partial x^j} \frac{\partial U^f}{\partial x^i} + \frac{\partial U^f}{\partial x^j} \frac{\partial U^m}{\partial x^i} \right) \right),$$

$$\begin{aligned}
\frac{\partial^2 N}{\partial x^i \partial p^k} &= \frac{\partial \left\{ g^m \gamma(\alpha) (g^f)^{\gamma(\alpha)-1} \right\}}{\partial p^k} \frac{\partial U^f}{\partial x^i} + \frac{\partial \left\{ (g^f)^{\gamma(\alpha)} \right\}}{\partial p^k} \frac{\partial U^m}{\partial x^i}, \\
\frac{\partial^2 N}{\partial x^i \partial y^h} &= \frac{\partial \left\{ g^m \gamma(\alpha) (g^f)^{\gamma(\alpha)-1} \right\}}{\partial y^h} \frac{\partial U^f}{\partial x^i} + \frac{\partial \left\{ (g^f)^{\gamma(\alpha)} \right\}}{\partial y^h} \frac{\partial U^m}{\partial x^i}, \\
\frac{\partial^2 N}{\partial x^i \partial V^f} &= \gamma(\gamma-1) \frac{g^m}{g^f} \frac{1}{g^f} (g^f)^\gamma (-1) \frac{\partial U^f}{\partial x^i} + \gamma \frac{1}{g^f} (g^f)^\gamma (-1) \frac{\partial U^m}{\partial x^i}, \\
\frac{\partial^2 N}{\partial x^i \partial V^m} &= \gamma \frac{1}{g^f} (g^f)^\gamma (-1) \frac{\partial U^f}{\partial x^i} + 0 \cdot \frac{\partial U^m}{\partial x^i}, \\
\frac{\partial^2 N}{\partial x^i \partial \gamma} &= \frac{g^m}{g^f} (g^f)^\gamma (1 + \gamma \ln g^f) \frac{\partial U^f}{\partial x^i} + (g^f)^{\gamma(\alpha)} \ln g^f \frac{\partial U^m}{\partial x^i},
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial \left\{ g^m \gamma(\alpha) (g^f)^{\gamma(\alpha)-1} \right\}}{\partial p^k} &= \left\{ \gamma(\gamma-1) \frac{g^m}{g^f} \frac{1}{g^f} (g^f)^\gamma (-1) \right\} \frac{\partial V^f}{\partial p^k} + \left\{ \gamma \frac{1}{g^f} (g^f)^\gamma (-1) \right\} \frac{\partial V^m}{\partial p^k} \\
&\quad + \left\{ \frac{g^m}{g^f} (g^f)^\gamma (1 + \gamma \ln g^f) \right\} \frac{\partial \gamma}{\partial p^k}, \forall k, \\
\frac{\partial \left\{ (g^f)^{\gamma(\alpha)} \right\}}{\partial p^k} &= \left\{ \gamma \frac{1}{g^f} (g^f)^\gamma (-1) \right\} \frac{\partial V^f}{\partial p^k} + \left\{ (g^f)^\gamma \ln g^f \right\} \frac{\partial \gamma}{\partial p^k}, \forall k, \\
\frac{\partial \left\{ g^m \gamma(\alpha) (g^f)^{\gamma(\alpha)-1} \right\}}{\partial y^h} &= \left\{ \gamma(\gamma-1) \frac{g^m}{g^f} \frac{1}{g^f} (g^f)^\gamma (-1) \right\} \frac{\partial V^f}{\partial y^h} + \left\{ \gamma \frac{1}{g^f} (g^f)^\gamma (-1) \right\} \frac{\partial V^m}{\partial y^h} \\
&\quad + \left\{ \frac{g^m}{g^f} (g^f)^\gamma (1 + \gamma \ln g^f) \right\} \frac{\partial \gamma}{\partial y^h}, \forall h, \\
\frac{\partial \left\{ (g^f)^{\gamma(\alpha)} \right\}}{\partial y^h} &= \left\{ \gamma \frac{1}{g^f} (g^f)^\gamma (-1) \right\} \frac{\partial V^f}{\partial y^h} + \left\{ (g^f)^\gamma \ln g^f \right\} \frac{\partial \gamma}{\partial y^h}, \forall h.
\end{aligned}$$

It is now easy to see that these equations can be arranged into the matrix equations (2.25) and (2.26).

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Chapter 3

Risk-Sharing Contracts under Repeated Double Moral Hazard

3.1 Background and Literature Review

In the simple, one-period model of *risk-sharing*, risk averse individuals are endowed with state-contingent incomes. These individuals enter into binding contracts which specify a transfer of income between individuals in each state of the world. Because individuals are risk averse, Pareto improving transfers exist and the set of equilibrium contracts is just the intersection of the set of Pareto efficient and the set of *ex ante* individually rational contracts. In such models with binding contracts, an efficient allocation of risk requires the marginal rate of substitution between state-contingent incomes to be equalized across individuals. When the game is finitely or infinitely repeated, this condition is simply required to hold each period.

When binding contracts are not available and the game is played a finite number of times, any equilibrium contract must also be *ex post* individually rational. While the initial endowment of state-contingent incomes may not be *ex ante* efficient, it will be the only *ex post* efficient allocation. This is because once the state of nature is observed, any transfer of income will raise one person's consumption only by reducing that of

the other. However in the absence of binding contracts, this must induce the latter to renege. Therefore when binding contracts are not available and the game is played a finite number of times, individuals will remain at their initial endowments since no equilibrium contracts exist.

In the absence of binding contracts, Pareto improving transfers may exist if we consider self-enforcing contracts, contracts with the property that no one ever wishes to renege *ex post*. The existence of such equilibria requires the game to be infinitely repeated (or at least that there is always a positive probability that the game will be played once more). The incentive never to renege once the state of nature is observed is achieved by imposing the requirement that at each period, any short-term gain from renegeing must be outweighed by the long-term benefits of cooperation. In this vein are the papers by Coate and Ravallion (1993), Kimball (1988), Ligon, Thomas and Worrall (1997) and Thomas and Worrall (1988) in which the authors characterize the set of equilibrium contracts that can be sustained in this way (i.e. the self-enforcing contracts).

These risk-sharing models, whether static or repeated and whether under binding or self-enforcing contracts, share a number of common features. First individuals are endowed with state-contingent incomes. This means that an individual's *ex post* income, the income he receives once a state has occurred, is completely determined by the occurrence of that state. More importantly it implies that there is no role for individual effort to influence the level of income received in a given state¹. Second, each individual's probability distribution over states (or incomes) is given². This means that an individual's *ex ante* income is also completely determined and once again there is no role for effort in influencing their probability distribution of income. Third, the income received by each individual (and the state of the world in which it is received) is common knowledge.

Allowing individual effort to affect the likelihood of obtaining different outcomes, with each person's effort being privately held information, leads to the well known *agency* prob-

¹ *Ex post*, different effort levels do not yield different incomes.

² These probabilities may be either the known probabilities attached to the occurrence of each state, or individuals' subjective probabilities.

lem (or *moral hazard* problem or *principal-agent* problem).³ In the standard (one-period) version of this problem (e.g. Holmström, 1979) a risk averse agent chooses a privately costly input (usually his own effort) into a production process. The output of the production process is uncertain and accrues to a risk neutral principal. Since the agent's effort is unobservable to the principal, and since it determines the likely occurrence of different output levels, the principal must choose a contract - consisting of a payment schedule and a particular effort level for the agent - that is both in the principal's best interest and would be accepted by the agent. The equilibrium contract must be *ex ante* individually rational and incentive compatible. This contract differs from the full-information first-best contract under which the agent's effort is fully observed by the principal. Under the first-best contract, a risk neutral principal assumes all the risk of the production process and makes a constant payment to the agent. Under a limited information contract, the agent is also required to bear some of the risk. This inefficiency reflects the conflict that exists between reducing the agent's risk and increasing his incentive to work for a higher rather than a lower return.

Individuals may take advantage of a *repeated* principal-agent relationship in order to reduce the inefficiency of short-term contracts. A repeated agency problem is one in which the one-period situation is repeated again and again, and there are several advantages of such long-term relationships. First, repetition of the one-period situation provides an opportunity to observe outcomes over a longer period of time and to improve one's inferences as to whether or not appropriate actions had been taken in the past. Second, a long term relationship creates the opportunity to reinforce the 'punishments' and 'rewards' that are offered for apparent deviations from the appropriate actions. Third, individuals' strategies can be designed to take account of past outcomes, i.e. of the history of the relationship. This creates additional opportunity for insuring against random

³Early work on moral hazard include Alchian and Demsetz (1972), Arrow (1963, 1965), Harris and Raviv (1976, 1978), Holmström (1979), Mirrlees (1974, 1975, 1976), Pauly (1974), Ross (1973), Shavell (1979), Simon (1951) and Wilson (1969). More recent contributions include Grossman and Hart (1983), Mayers and Smith (1981), Myerson (1983), Radner (1985) and Stiglitz (1983).

fluctuations in outcomes that were unrelated to actions, thereby reducing risk without harming incentives. Finally, in the theory of repeated games, conditions under which the noncooperative equilibria of the entire sequential game produce the cooperative outcomes of the component subgames can be identified and analyzed.

This final point was addressed in one of the early papers on repeated moral hazard. Radner (1981) examined a finitely repeated principal-agent relationship and showed that if the number of repetitions is sufficiently large (but finite), then there are approximate noncooperative equilibria⁴ of the entire repeated game that produce the cooperative outcome of each subgame. In particular, for any Pareto-optimal cooperative outcome of the one-period game that dominates the one-period noncooperative outcome, and any $\varepsilon > 0$, there exists for every sufficiently large T a noncooperative equilibrium of the T - period game that yields each player an average expected utility that is no less than his expected utility from the one-period cooperative game, minus ε . In order for noncooperative behaviour to produce cooperative outcomes in a repeated game, there must exist a statistical method for inferring the existence of inappropriate actions (when actions are not directly observed), and for doing so sufficiently rapidly to deter them. Furthermore, such a method must have a low probability of generating false alarms. Radner (1981) shows that such a method exists.

In a paper by Rubinstein and Yaari (1983), the authors move away from the common emphasis on rewards and penalties for inducing efficient outcomes. In the context of a simple (repeated) insurance problem, full-indemnity insurance contracts are offered at a pre-specified price. Hence the rewards and penalties incurred by the insured in any given period do not reflect the values taken by the observed variables in that period. However the price of the insurance contract can be changed over time to reflect the insured's past record. The authors show that as long as the insured is risk averse, introducing such flexibility into the temporal structure of rewards and penalties means that it is possible

⁴Approximate noncooperative equilibria in finite repetitions models is called an epsilon equilibrium if each player's strategy is within epsilon of being the best response to the other players' strategies.

to find an enforceable long-term contract that eliminates the inefficiency associated with moral hazard.

Radner (1985) focuses on the role of discounting in determining the effectiveness of the *threat* of future punishment as a deterrent to deviant behaviour. He points out that in a repeated principal-agent relationship, the players' strategies in any one repetition can be allowed to depend on the previous history of the game. This gives the principal an opportunity to observe the results of the agent's actions over several periods, to use statistical inference to 'establish' whether or not the agent had been choosing the appropriate actions, and to use the threat of punishment to deter the agent from deviant behaviour. However the effectiveness of such deterrents depends on the extent to which the agent discounts the future. Since the accumulation of reliable statistical evidence takes time, the threat of future punishment becomes a less effective deterrent the more the agent discounts the future. Radner (1985) demonstrates formally that in an infinitely repeated game, the less the players discount future utility the more they can approach the one-period efficient outcome with equilibria of the supergame. At the limit when there is no discounting, players can approximate the one-period efficient outcome with equilibria of the supergame.

In a finitely repeated agency relationship where both principal and agent discount the future, Lambert (1983) demonstrates that each period's sharing rule can be expressed in the same form as the sharing rule obtained by Holmström (1979) for the one-period model. He also shows that each period, both the agent's current compensation as well as his utility next period are increasing functions of the first period cash flow. This has the interpretation that the principal uses not only current but also future incentives to motivate the agent's effort in any given period. Finally, Lambert shows that the agent's compensation each period depends not only on his performance that period but also on his performance in previous periods. Specifically it is shown that the expected ratio of the principal's to the agent's marginal utility, in any period, must equal the known principal-agent marginal utility ratio in the previous period. One interpretation of this result

offered by Lambert is that the optimal long-term contract tries to smooth the agent's consumption over time. An alternative interpretation is that the principal spreads the risk of the first period outcome over as many periods as possible.

Rogerson (1985a) examines a finitely repeated principal-agent relationship in which the principal is risk neutral and the agent is risk averse. The principal is assumed to be able to borrow and save at a fixed interest rate but the agent has no access to the credit market. Also the agent discounts future consumption. The author shows that under the optimal contract, a simple relationship must hold between the wages offered in any two adjacent periods. In particular, the inverse of the agent's marginal utility of income, evaluated at any wage, must equal the expected value of the inverse of next period's marginal utility of income, conditioned upon last period's wage. Based on this result Rogerson offers the intuition that a repeated incentive problem provides the opportunity for intertemporal consumption smoothing and that the optimal contract will always take advantage of this. Also memory plays an important role in the optimal contract in that whenever an outcome affects the current wage it also affects the future period's wages. Another implication of Rogerson's result is that because of the incentive problem, the agent does not achieve the optimal level of consumption smoothing and is thus left with a residual desire to intertemporally self-insure through the use of credit markets. Therefore restriction of the agent's access to credit markets is necessary to achieve the Pareto-optimal outcome.

Thomas and Worrall (1990) apply the repeated incentive problem to examine the behaviour of debt/credit contracts over time. In a repeated unobserved endowment economy with legally enforceable loan contracts, the income of a single risk averse borrower (the agent) cannot be observed by a single risk neutral lender (the principal). In this setting the authors examine how debt contracts can be used to stabilize consumption for any finite or infinite time horizon and any discount factor between zero and one. It is shown that under the second-best contract, in order to reduce the cost of inducing incentive compatibility, it is necessary to reduce future utility by increasing future consumption

variation. Specifically it is shown that if the time horizon is infinite, the borrower's future utility becomes arbitrarily negative (with probability one). The borrower gets deeper and deeper into debt and consumption falls as debt increases. Despite this the authors show that it is possible to approach the first-best constant-consumption contract. In particular they demonstrate that as the discount factor tends to one and the time horizon tends to infinity, the second-best Pareto frontier converges pointwise to the first-best frontier.

This 'immiserization' result obtained by Thomas and Worrall (1990) is interesting and was also obtained by Green (1987) who showed that with a continuum of agents associated with an incentive problem (unobserved income endowments), the efficient societal arrangement involves almost all agents' consumption diverging to negative infinity. Although the papers by Green (1987) and Thomas and Worrall (1990) would seem to suggest that there is something about repeated moral hazard that ultimately causes a reduction in the utility/consumption of agents associated with an incentive problem, later papers cast doubt upon such an interpretation. Under different assumptions it is possible to create environments where the agent's consumption distribution does not exhibit a negative drift (Atkeson and Lucas, 1992), or their consumption distributions converge to a bounded limiting distribution (Wang, 1995). However as Phelan (1998) points out, it is difficult to identify, based on these papers, which characteristics of the moral hazard environment are relevant in determining the long run consumption of the agents. Phelan addresses this issue by isolating the roles of the different assumptions made in these models and arguing that it is the characteristics of the utility function that is the key determinant of agent's long run consumption.

As we have mentioned earlier and is now well known, a key feature of contracts formed in a repeated relationship is the dependence of agents' strategies on the previous history of outcomes (see Lambert, 1983; and Rogerson, 1985a). Such *non-stationary* strategies are in contrast to *stationary* strategies under which the action prescribed at time t depends only on the time t outcome. However the analytical intractabilities associated with history-dependent strategies often mean that it is very difficult to discern

important qualitative features of equilibrium contracts when these contracts are formed in repeated relationships. Spear and Srivastava (1987) address this issue by using recursive methods to reduce the dynamic problem of characterizing equilibrium contracts to a simple static problem in the calculus of variations. Their reformulated static problem is equivalent to the original dynamic problem but is much simpler to solve. It enables the equilibrium contract to be completely described by a set of functions that define a stationary Markovian solution, in which the expected discounted utility offered to the agent under the contract, conditioned on the previous history of outcomes, serves as the state variable.

Spear and Srivastava use this approach to analyze the optimal contract emerging from a standard repeated moral hazard problem. Under the optimal contract there is a critical level of output each period which, if exactly realized by the agent that period, means that next period's contract will be identical to the contract offered this period. However if the agent performs well (badly) this period by achieving an output that is greater (lower) than the critical output, then the agent is rewarded (punished) with a higher (lower) consumption today as well as a higher (lower) expected discounted utility payoff tomorrow. Furthermore if tomorrow's output is at its critical value, then the compensation paid tomorrow will be exactly what was paid today, and so forth. In this setting understanding the evolution of the contract requires an understanding of how the critical output level changes over time. However in their model it was not possible to unambiguously predict the direction of such changes.

We now return to our discussion of one-period agency problems but change our focus towards a discussion of *double moral hazard* (or *double agency*) problems, a situation in which the principal also makes choices that affect the likelihood of various outcomes. Several papers analyze economic problems that arise from a situation of double moral hazard and offer characterizations of the optimal contract under such circumstances. One of the key features of optimal contracts under double moral hazard is that the agent's reward is less sensitive to outcomes than would have been the case under a single agency

problem. When there is double moral hazard, outcomes are influenced by the actions of *both* principal and agent and so changes in outcomes convey only partial information about the agent's hidden actions.

In situations where both the principal and the agent are risk neutral, Bhattacharyya and Lafontaine (1995) and Romano (1994) show that the optimal contract is a simple linear contract in which both principal and agent proportionally share the output after a certain amount of transfer is made between them. However when the principal is risk neutral and the agent is risk averse, Kim and Wang (1998) show that the optimal contract is generally not linear. Also as the agent's risk aversion approaches zero (i.e. as the agent becomes less risk averse and more risk neutral), they show that the optimal contract does not approach a simple linear contract. The authors interpret this result as implying that under double moral hazard linear contracts are not a good approximation of the optimal contract when the agent is almost risk neutral.

There are several economic applications for the problem of double moral hazard. In the context of a firm with a risk neutral owner (the principal) and a risk averse worker (the agent), Demski and Sappington (1991) show that the double agency problem can be completely and costlessly resolved if the principal (who can observe but cannot prove that the agent shirks) has the option of requiring the agent to purchase the enterprise at a prenegotiated price. Such buyouts mean that the worker would become the residual claimant for the firm's stream of profit. The threat of having to purchase an enterprise whose value has been diminished by his own shirking provides the incentive for the worker to choose the efficient level of effort. Furthermore, knowing that the worker has worked diligently, the owner is also motivated to do likewise, rather than to shirk and transfer the enterprise to the worker. In their model it is the *threat* rather than the *exercise* of the buyout option that induces both parties to act efficiently.

Another application of the double moral hazard approach is to the analysis of warranties. Here both the producer and the consumer of a product take privately observed actions that affect the failure rate of the product. A producer can save costs by choosing

shoddy materials, thereby producing a less durable product. Furthermore product durability is only indirectly observed by consumers via the failure rate. On the other hand consumers can benefit by neglecting to maintain the product, thereby increasing failures. Also the consumer's actions in this regard are unobservable to the producer. A warranty is a payment from the producer to the consumer in the event of a breakdown. Such warranties increase the producer's incentives for durability and decrease the consumer's incentives for maintenance, hence the double moral hazard problem.

Papers that have analyzed the optimal warranty under such circumstances include Cooper and Ross (1985, 1988), Dybvig and Lutz (1993), Emons (1988) and Mann and Wissink (1988). Emons (1988) identified circumstances under which warranties solve the classic 'lemons' problem and induce firms to offer high-quality products. Therefore Shapiro's (1983) claim that warranties do not serve as a quality-assuring mechanism is not always valid. Using a continuous time model, Dybvig and Lutz (1993) showed that the optimal warranty is a block warranty that concentrates all payments as early as possible. Also if there is no upper bound on the size of the warranty payment, then the first-best outcome can be approximated by offering very large warranties for a very short period of time. However in equilibrium the maximum warranty may be bounded by the exogenously determined repair cost: if warranty payments are greater than the cost of repair, consumers have an incentive to repeatedly abuse the product and pocket the difference between the warranty payment and the repair cost. Cooper and Ross (1988) show that under certain conditions, a two-period warranty can induce the full-information first-best outcome. Finally Mann and Wissink (1988) show that efficient contracts exist when uncertainty is moderate, while inefficiencies result from either excessive or insufficient uncertainty.

Other applications of double moral hazard include the analysis of optimal contracts in agriculture (see Agrawal, 1999; Bhattacharyya and Lafontaine, 1995; Eswaran and Kotwal, 1985), and the analysis of royalty contracts in franchising (see Lafontaine and Shaw, 1996; Lutz, 1995; Mathewson and Winter, 1985; Rubin, 1978).

In this chapter we go beyond existing models of risk-sharing and of (repeated) moral hazard and examine a problem of risk-sharing under *repeated double moral hazard*. Specifically, we consider two risk averse individuals, labelled 1 and 2, involved in an infinitely repeated relationship and who both discount the future by a common discount factor. At each period, each agent chooses an effort level and then receives a stochastic income drawn from a time-invariant income distribution contingent on their own choice of effort. The income received by each agent is common knowledge. However the effort choice of each agent is private information. As with standard agency models, the problem is set up to ensure that neither agent's effort choice can be inferred with certainty from the observed income levels. At each period, once incomes are observed, a transfer of income from agent 2 to agent 1 is made and the game moves on to the next period. (Throughout the chapter we sometimes refer to agent 1 as the principal and to agent 2 as the agent. Nevertheless the model is perfectly symmetric with respect to both individuals and these individuals are identical). We assume that contracts are binding and so the problem of *ex post* rationality is not treated. There are no opportunities for individuals to borrow from or lend to outside parties, and so they are constrained to consume the aggregate endowment received each period. This generalized framework allows us to analyze how the optimal long-term contract deals with the trade-off between an efficient level of risk-sharing and the simultaneous provision of *two* sets of incentives. It is the analysis of these trade-offs, in the context of an infinitely repeated agency problem, that represents the key departure of this chapter from the existing literature.

Throughout our analysis we adopt the recursive methods used by Spear and Srivastava (1987) in order to establish the key qualitative features of the optimal contract under repeated double moral hazard. In a similar vein to their paper, suppose agent 2 is promised utility level w at time t . Two functions $a_1(w)$ and $a_2(w)$ specify, respectively, the effort level of agent 1 and agent 2 at t . Once time t income levels (y_1, y_2) are observed, a function $\tau(w, y_1, y_2)$ determines the time t transfer from 2 to 1 and a function $V(w, y_1, y_2)$ determines the level of utility promised to agent 2 at $t + 1$.

Both the transfer and the utility promise schedules depend on the incomes observed at t . Also the function $U(w)$ determines the payoff to agent 1 at t . Therefore at time $t + 1$, if time t incomes were (y_{1t}, y_{2t}) , agent 2 receives $w_{t+1} = V(w_t, y_{1t}, y_{2t})$ and individuals 1 and 2 work $a_1(w_{t+1})$ and $a_2(w_{t+1})$, respectively. If time $t + 1$ incomes are (y_{1t+1}, y_{2t+1}) , a $t + 1$ transfer of $\tau(w_{t+1}, y_{1t+1}, y_{2t+1})$ occurs from 2 to 1 and agent 2 is promised $w_{t+2} = V(w_{t+1}, y_{1t+1}, y_{2t+1})$ at $t + 2$. Agent 1 receives $U(w_{t+1})$ at $t + 1$, and so on.

In our repeated double moral hazard problem, the transfer and the utility promise schedules depend upon the incomes of *both* individuals. As a result, individuals' lifetime expected discounted utilities must be defined with respect to a *joint* (income) probability density function contingent upon the effort choices of *both* individuals. This implies that our recursive reduction of the problem yields a double integral version of the calculus of variations problem, which contrasts with the single integral version applicable in Spear and Srivastava (1987).

In characterizing the optimal contract under repeated double moral hazard, we examine both the contract's within-period characteristics as well as its evolution from one period to the next. Although we do not offer a full characterization of the optimal contract, several interesting properties emerge. First, a key feature of the optimal long-term contract under double moral hazard is the way in which this contract handles the trade-off between an efficient level of risk-sharing, on the one hand, and the simultaneous provision of *two* sets of incentives, on the other. Because a single transfer schedule must offer incentives to two individuals, such a contract is less sensitive to the performance of any single individual than would have been the case if there were only one person with an incentive problem. This result reiterates the predictions of one-period double agency models, however we show that in the infinitely repeated framework, it holds not only for the within-period incentives on offer, but also for the incentives that are offered over time.

Second, a well-known condition describing the optimal level of intertemporal con-

sumption smoothing under repeated single agency is generalized to take account of the double incentive problem. In the literature on repeated single agency (Lambert, 1983; Rogerson, 1985a; and Spear and Srivastava, 1987) it is shown that the *expected* ratio of person i 's to person j 's marginal utility in any period must always equal the *known* ratio of person i 's to person j 's marginal utility in the previous period, where $i, j = 1, 2$, $i \neq j$, and only j faces a binding incentive constraint. However when the single agency setting is generalized to take account of the double incentive problem, under the optimal contract there is some deviation from the level of intertemporal consumption smoothing that would have been optimal under repeated single agency. We show that this deviation can be characterized in terms of its benefits and costs. We also show that when *both* individuals face binding incentive constraints then the *expectation* of the ratio of person i 's to person j 's marginal utility in period t must always be strictly greater than the *known* ratio of person i 's to person j 's marginal utility in period $t - 1$, $i, j = 1, 2$, $i \neq j$.

We establish a number of other characteristics of the optimal contract. Although different income realizations have an ambiguous effect on the size of the income transfer between principal and agent, we show that *both* agents have current consumption and future utility payoffs that are monotonically increasing in their own, current level of income. Therefore *ceteris paribus*, a higher income realized by agent i at time t is rewarded by a higher consumption for i at t , as well as a higher utility payoff for i at $t + 1$ ($i = 1, 2$). We also show that there exists a unique *pair* of incomes, $(\tilde{y}_1(w), \tilde{y}_2(w))$, each period for which the agent's utility payoff (and hence the contract) is the same tomorrow as it is today. Furthermore the transfer and the principal-agent marginal utility ratio, evaluated at these critical incomes, can be thought of as 'first-best'. These results generalize those obtained by Spear and Srivastava (1987) in their analysis of a repeated single agency problem.

Note that the critical income, $\tilde{y}_i(w)$, offers a benchmark level of income against which person i 's performance can be judged. Suppose i ' income is y_i . Then $y_i > \tilde{y}_i$ can be interpreted as saying that person i has performed 'relatively well', while $y_i < \tilde{y}_i$ implies

that person i 's performance has been 'relatively poor'.

Combining earlier results we show that whenever the principal performs well and/or the agent performs badly in the current period, then tomorrow the principal is rewarded (the agent punished) with a higher (lower) utility payoff than the payoffs received today. Also the current transfer is chosen so that the current principal-agent marginal utility ratio is low relative to the first-best marginal utility ratio. The opposite is true if the principal performs badly and/or the agent performs well in the current period. However if *both* principal and agent perform well (or if both perform badly) in the current period, then the effect of this on the current principal-agent marginal utility ratio *vis-a-vis* the first-best ratio, and on the future utility payoffs *vis-a-vis* the current payoffs, is ambiguous. We interpret this as meaning that the optimal contract, in terms of the structure of rewards and penalties offered within any given period, is less sensitive to the performance of any single individual than would have been the case under a repeated single agency problem. Remember that under repeated single agency, a good performance by the agent is always rewarded (and a bad performance is *always* punished) under the optimal contract. Furthermore these rewards and punishments are entirely unrelated to the performance of any other individual.

We also examine the evolution of the contract over time. Specifically we look at the relationship between the time $t + 1$ and the time t marginal utility ratios, and between the time $t + 2$ and the time $t + 1$ utility payoffs. We show that if, at $t + 1$, both agents obtain their $t + 1$ critical incomes, then the $t + 1$ marginal utility ratio will be the same as the known marginal utility ratio at t , and the $t + 2$ utility payoffs will be the same as the $t + 1$ utility payoffs. If, however, at $t + 1$ the principal 'performs badly' and/or the agent 'performs well' (i.e. performance being judged relative to the $t + 1$ critical incomes), then the $t + 1$ principal-agent marginal utility ratio will be greater than the known principal-agent marginal utility ratio at t , and the principal's (agent's) $t + 2$ utility payoff will also be lower (higher) than his $t + 1$ utility payoff. The opposite is true if at time $t + 1$, the principal 'performs well' and/or the agent 'performs badly'. Finally, if *both*

principal and agent perform well (or if they *both* perform badly), then we cannot predict the effect of this on the $t + 1$ marginal utility ratio *vis-a-vis* the time t ratio, nor on the $t + 2$ utility payoffs *vis-a-vis* the $t + 1$ payoffs. Here we see that the optimal contract, this time in terms of the structure of rewards and penalties offered over time, continues to be less sensitive to the performance of any single individual than would have been the case under repeated single agency.

This chapter is structured as follows. In section 3.2 we set up the model and in section 3.3 we specify the dynamic problem of choosing the optimal contract under repeated double moral hazard. In section 3.4 we adopt the approach of Spear and Srivastava (1987) in reducing this problem to a static problem in the calculus of variations. In section 3.5 we obtain features of the full-information first-best contract, i.e. the efficient contract, under which agents' chosen effort levels are fully observable. This provides a benchmark for later comparison with the second-best contract, i.e. the optimal contract under repeated double moral hazard. In this section we present the within-period characterization of the first-best contract as well as a characterization of its evolution over time. In section 3.6 the second-best contract is analyzed. As with the first-best contract we examine the main features of this contract within any given time period and discuss the contract's evolution over time. Section 3.7 concludes and suggests areas for further work.

3.2 The Model

There is an infinite sequence of dates, $t \in T = \{1, 2, \dots\}$, and two infinitely-lived agents, $i = 1, 2$. Agents are risk averse and maximize their expected lifetime utilities. They discount the future by the common discount rate $\beta \in (0, 1)$. Although both agents are identical, we will refer to 1 as the principal and 2 as the agent.

At each $t \in T$, agent i chooses an effort level $a_{it} \in A = [\underline{a}, \bar{a}] \subset R_+$ and receives a stochastic income $y_{it} \in [0, \bar{y}]$, $\bar{y} > 0$, drawn from his effort-contingent distribution $F_i(y_i, a_i) : R \rightarrow [0, 1]$. It is assumed that $F_i(y_i, a_i)$ has a density denoted by $f_i(y_i, a_i)$,

with f_{ia_i} and $f_{ia_ia_i}$ well defined for all (y_i, a_i) . The *joint* (effort-contingent) distribution of incomes at each t is given by $F(y_1, y_2, a_1, a_2) : R^2 \rightarrow [0, 1]$. We assume that the *joint* density $f(y_1, y_2, a_1, a_2)$ exists, with f_{a_i} and $f_{a_ia_j}$ ($i, j = 1, 2$) well defined for all (y_1, y_2, a_1, a_2) .

Throughout we make the following assumptions.

Assumption 1. Each agent observes his own chosen effort level but not that of the other.

Assumption 2. Each agent's income is observed by both agents.

Assumption 3. At any time t , individual incomes y_1 and y_2 are independent random variables. Therefore $f(y_1, y_2, a_1, a_2) = f_1(y_1, a_1) f_2(y_2, a_2)$ and $F(y_1, y_2, a_1, a_2) = F_1(y_1, a_1) F_2(y_2, a_2)$.

Assumption 4 (stochastic dominance). A change in individual i 's effort alters their distribution of income $F_i(y_i, a_i)$ such that $F_{ia_i}(y_i, a_i) \leq 0$ for all $y_i \in R$, and $F_{ia_i}(y_i, a_i) < 0$ for some y_i values, $i = 1, 2$. Also a change in individual i 's effort has no impact on j 's distribution of income: $F_{ja_i}(y_j, a_j) = 0$, all $y_j \in R, i \neq j$.

Stochastic dominance means that more effort generally reduces the likelihood of receiving low incomes. In the standard single agency problem this assumption is both necessary and sufficient for increases in the agent's effort to make the principal better off. However under repeated double moral hazard this is not the case. Nevertheless we see that under stochastic dominance, increases in a_i make person i 's distribution function more favourable to person j ($i, j = 1, 2, i \neq j$) and so we adopt this assumption in the current setting. Assumption 4 also states that one person's effort does not affect another person's income distribution.

Assumption 5 (the convex distribution function condition). $F_{ia_ia_i}(y_i, a_i) \geq 0$ for all $y_i \in R, i = 1, 2$.

Assumption 5 has the implication that although i 's distribution function becomes increasingly favourable to j as a_i increases, it does so at a decreasing rate. Hence there

are stochastically decreasing marginal returns to increases in effort.

Assumption 6 (the monotone likelihood ratio condition).

$$\frac{\partial}{\partial y_i} \left(\frac{f_{ia_i}(y_i, a_i(w))}{f_i(y_i, a_i(w))} \right) > 0, i = 1, 2.$$

Assumption 6 implies that greater effort will reduce the likelihood of poor outcomes and increase the likelihood of better outcomes. This interpretation will become clearer when we obtain lemma 1, below.

In the standard single agency problem, the convex distribution function condition (*cdfc*) combined with the monotone likelihood ratio condition (*mlrc*) imply that both the principal and the agent have objective functions that are concave in the agent's effort. Hence the problem of choosing effort to maximize either person's objective, or to maximize a positively weighted sum of the two objectives, is well behaved. Specifically there is a global maximum that satisfies standard first order conditions. However under repeated double moral hazard this is not the case and so in our setting we will make the explicit assumption (assumption 9) that each individual's objective function is concave in (a_1, a_2) .

Assumption 7. The supports of the income distributions $F_i(y_i, a_i)$, $i = 1, 2$, and $F(y_1, y_2, a_1, a_2)$, given by $[0, \bar{y}]$ and $[0, \bar{y}]^2$ respectively, are compact and do not alter with changes in either agent's effort.

Assumption 7 means that although, at any date t , i 's income distribution F_i varies as a_{it} alters, the actual income received by i must always belong to the set $[0, \bar{y}]$, irrespective of i 's chosen effort level. Therefore from any observed income it is not possible to infer with certainty the effort level chosen by i . This assumption also ensures that neither individual can force the other (by levying a sufficiently large punishment) to choose a particular level of effort. If this were possible there would effectively be no agency problem.

Assumption 8. For any $a_i \in A$ ($i = 1, 2$) individual incomes are *independently and identically distributed* over time.

Assumption 8 means that for the same choice of a_i each period, income distributions are both time invariant and are independent over time.

The sequencing of observations and choices are as follows. At each date t , each agent chooses an effort level $a_{it} \in A$. After period t incomes are observed, a transfer of income equal to τ_t from agent 2 to agent 1 is made and the game moves on to time $t + 1$.

The transfer at any date t can depend on any variables that are jointly observable at the time the transfer is made. Since the transfer at t is made after period t incomes are observed, the transfer at t can depend on the entire history of incomes, up to and including incomes at t . Although the transfer at t can also depend on the history of transfers to $t - 1$, note that the time 1 transfer depends only on time 1 incomes. Therefore by solving out recursively, the transfer at t will ultimately depend only on current and past incomes.

Let $y^t = \{y_1, \dots, y_t\}$, where y_t denotes a vector of incomes (y_{1t}, y_{2t}) , represent the history of incomes up to and including t . Then the transfer at t can be written $\tau_t(y^t)$.

Note that the function $\tau_t(y^t)$ is parameterized by the history of incomes to $t - 1$, namely y^{t-1} . Therefore at time t individuals face a different time t transfer schedule for different histories y^{t-1} . It may therefore be that different histories y^{t-1} imply different optimal effort levels at t . Furthermore since time t effort levels are chosen before time t income levels are observed, effort at t cannot depend on the incomes realized at t . We therefore condition each agent's effort level at t on the history of incomes to $t - 1$. Once again conditioning current effort levels on past transfers makes no difference to this specification since solving out recursively shows that time t effort levels will depend only on y^{t-1} . We therefore write $a_{it}(y^{t-1})$ ($i = 1, 2$) as agent i 's effort level at time t . When $t = 1$, we write $a_{i1}(y^0) = a_{i1}$, where y^0 refers to the "empty history" $\{\}$.

A strategy at t is therefore given by the functions $\tau_t(y^t)$ and $a_{it}(y^{t-1})$, $i = 1, 2$. Note that we do not condition i 's effort level at t on i 's previous history of effort levels. Since we will assume that each person's utility is separable over time, individuals do not need

to take account of past effort choices when choosing their current level of effort.

The time-invariant *ex-post* utility functions of the principal and agent are additively separable in time t own consumption and effort and are given, respectively, by

$$u(y_{1t} + \tau_t(y^t)) - a_{1t}(y^{t-1})$$

and

$$v(y_{2t} - \tau_t(y^t)) - a_{2t}(y^{t-1}),$$

where $u, v : C \rightarrow R$, $C = [0, 2\bar{y}]$, with $u_c, v_c > 0$ and $u_{cc}, v_{cc} < 0$, for all $c \in C$.

3.3 The Constrained-Efficiency (Second-Best) Problem

Definition 1. Let τ represent the sequence of strategies $\{\tau_t(y^t)\}_{t=1}^{\infty}$, and a_i the sequence of strategies $\{a_{it}(y^{t-1})\}_{t=1}^{\infty}$ ($i = 1, 2$).⁵ Then a *risk-sharing contract* is the sequence of strategies $\{\tau, a_1, a_2\}$ given by

$$\{\tau_t(y^t), a_{1t}(y^{t-1}), a_{2t}(y^{t-1})\}_{t=1}^{\infty}.$$

Definition 2. Given any history of incomes y^t , the *stream of expected utilities* from $t + 1$ onwards, discounted to $t + 1$, from a contract $\{\tau, a_1, a_2\}$, is given by

$$\begin{aligned} & U(y^t, \tau, a_1, a_2) \\ &= \sum_{s=1}^{\infty} \beta^{s-1} \int \dots \int [u(y_{1t+s} + \tau_{t+s}(y^{t+s})) - a_{1t+s}(y^{t+s-1})] d^{2s} F(y^{t+s}; y^t, a_1, a_2) \end{aligned}$$

⁵From now on we use the notation a_i to represent both a sequence of effort choices by individual i , such as $\{a_{it}(y^{t-1})\}_{t=1}^{\infty}$, as well as a single action in $F_i(y, a_i)$ or in $F(y_1, y_2, a_1, a_2)$. Which meaning is intended will be clear from the context.

(3.1)

for the principal, and

$$\begin{aligned}
 & V(y^t, \tau, a_1, a_2) \\
 &= \sum_{s=1}^{\infty} \beta^{s-1} \int \dots \int [v(y_{2t+s} - \tau_{t+s}(y^{t+s})) - a_{2t+s}(y^{t+s-1})] d^{2s} F(y^{t+s}; y^t, a_1, a_2)
 \end{aligned}
 \tag{3.2}$$

for the agent, where β is the common discount factor and $F(y^{t+s}; y^t, a_1, a_2)$ is a distribution function defined on the space of finite histories (i.e. finite sequences of pairs of incomes) to $t + s$ ($s = 1, 2, \dots$) for a given history to t , and contingent upon a sequence of chosen effort levels. The stream of expected utilities from $t = 1$ onwards, discounted back to time 1, can therefore be written as $U(\tau, a_1, a_2)$ for the principal, and $V(\tau, a_1, a_2)$ for the agent.

Definition 3. A contract $\{\tau, a_1, a_2\}$ is *feasible* if, for every y^t , $\tau_t(y^t) \in [-y_{1t}, y_{2t}]$, and for every y^{t-1} , $a_{it}(y^{t-1}) \in A$, $i = 1, 2$.

Definition 4. Let \tilde{a}_i represent the particular strategy sequence $\{\tilde{a}_{it}(y^{t-1})\}_{t=1}^{\infty}$ ($i = 1, 2$). Then a contract $\{\tau, a_1, a_2\}$ is *incentive compatible* if, for every y^t ,

$$U(y^t, \tau, a_1, a_2) \geq U(y^t, \tau, \tilde{a}_1, a_2), \tag{3.3}$$

for all feasible strategy sequences \tilde{a}_1 , and

$$V(y^t, \tau, a_1, a_2) \geq V(y^t, \tau, a_1, \tilde{a}_2), \tag{3.4}$$

for all feasible strategy sequences \tilde{a}_2 . This requires

$$\{a_{1t+s}(y^{t+s-1})\}_{s=1}^{\infty} \in \arg \max_{\{\tilde{a}_{1t+s}(y^{t+s-1})\}_{s=1}^{\infty} \in A} U(y^t, \tau, \tilde{a}_1, a_2), \text{ given } \tau, a_2,$$

and

$$\{a_{2t+s}(y^{t+s-1})\}_{s=1}^{\infty} \in \arg \max_{\{\tilde{a}_{2t+s}(y^{t+s-1})\}_{s=1}^{\infty} \in A} V(y^t, \tau, a_1, \tilde{a}_2), \text{ given } \tau, a_1,$$

for every y^t .

Definition 5. A contract $\{\tau, a_1, a_2\}$ is *optimal* (or *constrained-efficient* or *second-best*) if, from $t = 1$ onwards, $\{\tau, a_1, a_2\}$ maximizes $U(\tau, a_1, a_2)$ subject to feasibility, incentive compatibility, and subject to the agent receiving the expected discounted utility payoff

$$V(\tau, a_1, a_2) = w, \tag{3.5}$$

where w could be the outcome of some (unspecified) negotiation process.

Our definition of the optimal contract assumes that all contracts are binding and enforceable so that it is impossible for either the principal or the agent to renege *ex post*. The requirement that contracts satisfy the double incentive compatibility constraints reflects the fact that neither person can observe the level of effort chosen by their partner.

3.4 Reduction of the Problem of Choosing the Second-Best Contract

In this section we follow Spear and Srivastava (1987, p. 603) in reducing the problem of finding the optimal contract to a static problem in the calculus of variations. First, note that by applying recursive methods and starting from $t = 1$, conditions (3.1) and (3.2) may be re-written as

$$\begin{aligned} U(\tau, a_1, a_2) = & \int \int_{[0, \bar{y}]^2} u(y_{11} + \tau_1(y^1)) f(y_{11}, y_{21}, a_{11}, a_{21}) dy_{11} dy_{21} \\ & + \beta \int \int_{[0, \bar{y}]^2} U(y^1, \tau, a_1, a_2) f(y_{11}, y_{21}, a_{11}, a_{21}) dy_{11} dy_{21} \\ & - a_{11} \end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
V(\tau, a_1, a_2) = & \int \int_{[0, \bar{y}]^2} v(y_{21} - \tau_1(y^1)) f(y_{11}, y_{21}, a_{11}, a_{21}) dy_{11} dy_{21} \\
& + \beta \int \int_{[0, \bar{y}]^2} V(y^1, \tau, a_1, a_2) f(y_{11}, y_{21}, a_{11}, a_{21}) dy_{11} dy_{21} \\
& - a_{21},
\end{aligned} \tag{3.7}$$

respectively, where $f(y_{11}, y_{21}, a_{11}, a_{21})$ is the joint probability density over time 1 incomes contingent on time 1 effort levels. (See the Appendix to this chapter for a detailed derivation of (3.6) and (3.7)). Suppose $\{\tau, a_1, a_2\}$ is the optimal contract. Then, subject to feasibility and incentive compatibility, $U(y^1, \tau, a_1, a_2)$ must be the maximum expected discounted utility received by the principal given that the agent receives $V(y^1, \tau, a_1, a_2)$. The proof of this statement is straightforward. If it were not true we would be able to replace the portion of $\{\tau, a_1, a_2\}$ that corresponds to the subgame beginning in period 2 with a new contract that maximizes $U(y^1, \tau, a_1, a_2)$ subject to the agent receiving $V(y^1, \tau, a_1, a_2)$, and subject to feasibility and incentive compatibility. But this would mean that the new contract, starting from $t = 1$, would be incentive compatible (note that the new contract must relax the time 1 incentive compatibility constraint) and would be Pareto superior to $\{\tau, a_1, a_2\}$. It follows that the contract $\{\tau, a_1, a_2\}$ could not have been optimal in the first place.

Now let $\Lambda = \{V(y^t, \tau, a_1, a_2)\}_{t=0}^{\infty}$ and $\Psi = \{U(y^t, \tau, a_1, a_2)\}_{t=0}^{\infty}$, where these values are evaluated at the optimal contract. For any $w \in \Lambda$, consider the problem of choosing $\{\tau, a_1, a_2\}$ to maximize $U(\tau, a_1, a_2)$ subject to feasibility, incentive compatibility and $V(\tau, a_1, a_2) = w$. Hence $\{\tau, a_1, a_2\}$ maximizes the principal's stream of payoffs from $t = 1$ onwards, discounted back to time 1, subject to the agent receiving w , incentive compatibility, and feasibility.

In the solution to this problem and for any payoff w to the agent, let $U(w)$ represent the principal's time 1 payoff, $a_i(w)$ ($i = 1, 2$) the time 1 effort choices of the principal and agent, $\tau(w, y_1, y_2)$ the time 1 transfer if time 1 incomes are (y_1, y_2) , and $V(w, y_1, y_2)$

the time 2 utility payoff (i.e. the stream of payoffs from $t = 2$ onwards, discounted back to time 2) promised to the agent if time 1 incomes are (y_1, y_2) .

This suggests that if $w = V(\tau, a_1, a_2)$, then $U(w) = U(\tau, a_1, a_2)$ and $a_i(w) = a_{i1}(y^0) = a_{i1}$ ($i = 1, 2$). Also, if time 1 incomes are (y_1, y_2) , then $\tau(w, y_1, y_2) = \tau_1(y^1)$ and $V(w, y_1, y_2) = V(y^1, \tau, a_1, a_2)$. We also have $U(V(w, y_1, y_2)) = U(y^1, \tau, a_1, a_2)$ and $a_i(V(w, y_1, y_2)) = a_{i2}(y^1)$ ($i = 1, 2$). If time 2 incomes are (y'_1, y'_2) , then $\tau(V(w, y_1, y_2), y'_1, y'_2) = \tau_2(y^2)$ and $V(V(w, y_1, y_2), y'_1, y'_2) = V(y^2, \tau, a_1, a_2)$, etc.

In the solution to this problem therefore, the players' strategies, and hence the risk-sharing contract, will evolve as follows. At $t = 1$, the principal offers $\tau(w, y_1, y_2)$ and $V(w, y_1, y_2)$ as the transfer and promised future utility schedules, respectively. The principal and agent will choose effort levels $a_i(w)$ ($i = 1, 2$), which must be incentive compatible, and will receive payoffs $U(w)$ and w respectively. At $t = 2$, if (y_1, y_2) occurred at $t = 1$, the agent receives the utility payoff $V(w, y_1, y_2)$. In this case, the principal offers $\tau(V(w, y_1, y_2), y'_1, y'_2)$ and $V(V(w, y_1, y_2), y'_1, y'_2)$ as the transfer and promised future utility schedules respectively. The principal and agent will choose incentive compatible effort levels $a_i(V(w, y_1, y_2))$ ($i = 1, 2$) and receive payoffs $U(V(w, y_1, y_2))$ and $V(w, y_1, y_2)$ respectively. Note that at $t = 2$, $U(V(w, y_1, y_2))$ is the maximum utility the principal can receive given that the agent receives $V(w, y_1, y_2)$. At $t = 3$, if (y'_1, y'_2) occurred at $t = 2$, the agent receives the utility payoff $V(V(w, y_1, y_2), y'_1, y'_2)$. In this case, the principal offers $\tau(V(V(w, y_1, y_2), y'_1, y'_2), y''_1, y''_2)$ and $V(V(V(w, y_1, y_2), y'_1, y'_2), y''_1, y''_2)$, etc.

The preceding argument suggests that any solution to the problem of finding our optimal contract can be characterized by five functions

$$V(w, y_1, y_2) : \Lambda \times [0, \bar{y}]^2 \rightarrow \Lambda$$

$$U(w) : \Lambda \rightarrow \Psi$$

$$\tau(w, y_1, y_2) : \Lambda \times [0, \bar{y}]^2 \rightarrow [-y_1, y_2]$$

and

$$a_i(w) : \Lambda \rightarrow A \quad (i = 1, 2)$$

that satisfy, for each $w \in \Lambda$,

$$\begin{aligned} w &= \int \int_{[0, \bar{y}]^2} \left[\begin{array}{c} v(y_2 - \tau(w, y_1, y_2)) \\ + \beta V(w, y_1, y_2) \end{array} \right] f(y_1, y_2, a_1(w), a_2(w)) dy_1 dy_2 - a_2(w) \\ &\geq \int \int_{[0, \bar{y}]^2} \left[\begin{array}{c} v(y_2 - \tau(w, y_1, y_2)) \\ + \beta V(w, y_1, y_2) \end{array} \right] f(y_1, y_2, a_1(w), a_2) dy_1 dy_2 - a_2, \forall a_2 \in A, \end{aligned} \quad (3.8)$$

$$\begin{aligned} U(w) &= \int \int_{[0, \bar{y}]^2} \left[\begin{array}{c} u(y_1 + \tau(w, y_1, y_2)) \\ + \beta U(V(w, y_1, y_2)) \end{array} \right] f(y_1, y_2, a_1(w), a_2(w)) dy_1 dy_2 - a_1(w) \\ &\geq \int \int_{[0, \bar{y}]^2} \left[\begin{array}{c} u(y_1 + \tau(w, y_1, y_2)) \\ + \beta U(V(w, y_1, y_2)) \end{array} \right] f(y_1, y_2, a_1, a_2(w)) dy_1 dy_2 - a_1, \forall a_1 \in A \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} w' &= V(w, y_1, y_2) \\ &= \int \int_{[0, \bar{y}]^2} \left[\begin{array}{c} v(y'_2 - \tau(w', y'_1, y'_2)) \\ + \beta V(w', y'_1, y'_2) \end{array} \right] f(y'_1, y'_2, a_1(w'), a_2(w')) dy'_1 dy'_2 - a_2(w'). \end{aligned} \quad (3.10)$$

Conditions (3.8) and (3.9) state that w is the agent's expected utility payoff, $U(w)$ is the principal's expected utility payoff, and that both $a_1(w)$ and $a_2(w)$ must be incentive compatible. Condition (3.10) tells us the relationship between the agent's payoff in any two consecutive periods. If the agent receives $V(w, y_1, y_2)$ today then he must be offered $V(V(w, y_1, y_2), y'_1, y'_2)$ tomorrow. Condition (3.9) possesses a similar interpretation for the principal.

We can now restate the problem of finding the optimal contract (definition 5). First, define

$$Eu \equiv \int \int_{[0,\bar{y}]^2} \begin{bmatrix} u(y_1 + \tau(w, y_1, y_2)) \\ +\beta U(V(w, y_1, y_2)) \end{bmatrix} f(y_1, y_2, a_1(w), a_2(w)) dy_1 dy_2 - a_1(w)$$

and

$$Ev \equiv \int \int_{[0,\bar{y}]^2} \begin{bmatrix} v(y_2 - \tau(w, y_1, y_2)) \\ +\beta V(w, y_1, y_2) \end{bmatrix} f(y_1, y_2, a_1(w), a_2(w)) dy_1 dy_2 - a_2(w)$$

as the expected discounted utilities of the principal and the agent, respectively, at any time t .

Definition 6. Assuming interior solutions for $a_1(w)$ and $a_2(w)$, a contract

$$\{\tau(w, y_1, y_2), V(w, y_1, y_2), a_1(w), a_2(w)\}$$

is optimal if, for each $w \in \Lambda$,

$$U(w) = \max_{\{\tau(w, y_1, y_2), V(w, y_1, y_2), a_1(w), a_2(w)\}} Eu \quad (P)$$

subject to $Ev = w$, $\frac{\partial Ev}{\partial a_2} = 0$, and $\frac{\partial Eu}{\partial a_1} = 0$.

Note that in this specification of the problem of finding the optimal contract we have replaced each person's incentive compatibility constraint by the first order condition for their problem of choosing effort (i.e. $\frac{\partial Eu}{\partial a_1} = 0$ for the principal and $\frac{\partial Ev}{\partial a_2} = 0$ for the agent).

In solving (P) we can first choose two intervals Λ and Ψ as follows. The smallest value of $w \in \Lambda$ could be the value for which, each period, the agent works as much as possible and consumes nothing. The largest value of $w \in \Lambda$ could be the value for which, each period, the agent works as little as possible and consumes the total realized income. Then the endpoints of Ψ would be the utility payoffs flowing to the principal, under the

optimal contract, at the above extreme values of Λ . Under any optimal contract therefore the agent would receive a payoff in Λ while the principal would receive a payoff in Ψ .

As w varies in Λ , the function $U(w)$ generates the Pareto frontier. We expect $U(w)$ to be non-increasing, i.e. $U'(w) \leq 0$, otherwise it would be possible to make the principal better off simply by offering the agent a higher expected discounted utility.

For any $w \in \Lambda$, the problem of choosing the schedules $\tau(w, y_1, y_2)$ and $V(w, y_1, y_2)$ is a problem in the calculus of variations. These functions are chosen from among functions that take specified values for each (y_1, y_2) on the boundary of $[0, \bar{y}]^2$. In the solution to (P), the functions $\tau(w, y_1, y_2)$, $V(w, y_1, y_2)$, $a_1(w)$, $a_2(w)$ and $U(w)$ describe a stationary Markovian solution in which w acts as the state variable.

In order to ensure that the problem (P) is well behaved, the following additional assumptions are required.

Assumption 9. For each $w \in \Lambda$, both Eu and Ev are concave in (a_1, a_2) .

Assumption 10. For each $w \in \Lambda$ and for effort levels $a_1(w)$ and $a_2(w)$, the integrands in Eu and Ev are twice continuously differentiable in their four *independent* arguments $y_1, y_2, \tau(w, y_1, y_2)$ and $V(w, y_1, y_2)$, and are jointly concave in $\tau(w, y_1, y_2)$ and $V(w, y_1, y_2)$.

Assumption 11. The functions $\tau(w, y_1, y_2)$ and $V(w, y_1, y_2)$ are twice continuously differentiable in y_1 and y_2 .

To ensure that the optimal contract is incentive compatible it is essential that the effort levels specified by the contract are those that would actually be chosen by the principal and the agent. Assumption 9 ensures that the first order approach of replacing i 's incentive compatibility condition by i 's first order condition for choice of a_i will be valid⁶. Under assumption 9 these first order conditions will be both necessary and sufficient for

⁶Several papers adopt this first-order approach of replacing an individual's incentive compatibility condition by his first order condition for choice of effort. See Spear and Srivastava (1987) and Lambert (1983) for examples. Also see Rogerson (1985b) and Jewitt (1988) for a discussion of the first-order approach to principal-agent problems.

the $a_i(w)$ offered under the optimal contract to yield a global maximum of i 's expected discounted utility ($i = 1, 2$).

Assumptions 10 and 11 are required to ensure that given a pair of effort levels $a_1(w)$ and $a_2(w)$, the calculus of variations problem of choosing the schedules $\tau(w, y_1, y_2)$ and $V(w, y_1, y_2)$ will be well behaved. In particular, any pair of surfaces which satisfy the appropriate boundary conditions as well as the simultaneous system of Euler-Lagrange equations will achieve a maximum.

3.5 A Benchmark: The Full-Information First-Best (Efficient) Contract

When both individuals' effort choices are fully observable, the principal simply has to solve (P), ignoring the incentive compatibility requirements. The resulting problem is

$$U(w) = \max_{\{\tau(w, y_1, y_2), V(w, y_1, y_2), a_1(w), a_2(w)\}} Eu \quad (P')$$

subject to $Ev = w, \forall w \in \Lambda$.

The solution to (P') yields the full-information first-best (or efficient) contract which we will denote by

$$\{\tau^*(w, y_1, y_2), V^*(w, y_1, y_2), a_1^*(w), a_2^*(w)\},$$

and which satisfies

$$\frac{u_{c_1}(y_1 + \tau^*(w, y_1, y_2))}{v_{c_2}(y_2 - \tau^*(w, y_1, y_2))} = -\lambda_1^*(w), \quad (3.11)$$

$$U'[V^*(w, y_1, y_2)] = \lambda_1^*(w), \quad (3.12)$$

$$\frac{\partial Eu}{\partial a_1} - \lambda_1^*(w) \frac{\partial Ev}{\partial a_1} = 0 \quad (3.13)$$

and

$$\frac{\partial Eu}{\partial a_2} - \lambda_1^*(w) \frac{\partial Ev}{\partial a_2} = 0, \quad (3.14)$$

where $\lambda_1^*(w)$ is the Lagrange multiplier on the agent's utility payoff constraint. Note that both (3.13) and (3.14) are evaluated at the full-information first-best contract.

3.5.1 The Efficient Contract Within a Given Time Period

Equations (3.11) - (3.14) provide conditions satisfied by

$$\{\tau^*(w, y_1, y_2), V^*(w, y_1, y_2), a_1^*(w), a_2^*(w)\}$$

at any given time. Equations (3.11) and (3.12) show that $\tau^*(w, y_1, y_2)$ and $V^*(w, y_1, y_2)$ are chosen to keep $\frac{u_{c1}(y_1 + \tau^*(w, y_1, y_2))}{v_{c2}(y_2 - \tau^*(w, y_1, y_2))}$ and $U'[V^*(w, y_1, y_2)]$ constant, irrespective of the incomes realized in any given period of time. Since the marginal utilities of consumption are positive, then we must have $\lambda_1^*(w) < 0$ (from (3.11)). This tells us that the utility promise constraint is binding and the agent receives exactly the utility payoff w at time t .

Also since $\lambda_1^*(w) < 0$, then equation (3.12) confirms that

$$U'[V^*(w, y_1, y_2)] < 0.$$

Recall that the function $U(w)$ generates the Pareto frontier as w varies in Λ . Then this confirms that the Pareto frontier is negatively sloped. Therefore under

$$\{\tau^*(w, y_1, y_2), V^*(w, y_1, y_2), a_1^*(w), a_2^*(w)\},$$

any increase in w must lead to a reduction in the maximized value of the utility payoff accruing to the principal.

From Borch (1962) and Holmström (1979) we know that the transfer $\tau^*(w, y_1, y_2)$

will be efficient from a risk-sharing point of view only if the right hand side of (3.11) is constant. Consider any pair of income levels (y_1, y_2) and $(\tilde{y}_1, \tilde{y}_2) \in [0, \bar{y}]^2$. Evaluating (3.11) at these income levels, eliminating $\lambda_1^*(w)$ from the resulting equations (since $\lambda_1^*(w)$ does not depend on incomes) and re-arranging the result yields

$$\frac{u_{c_1}(y_1 + \tau^*(w, y_1, y_2))}{u_{c_1}(\tilde{y}_1 + \tau^*(w, \tilde{y}_1, \tilde{y}_2))} = \frac{v_{c_2}(y_2 - \tau^*(w, y_1, y_2))}{v_{c_2}(\tilde{y}_2 - \tau^*(w, \tilde{y}_1, \tilde{y}_2))}. \quad (3.15)$$

If (y_1, y_2) and $(\tilde{y}_1, \tilde{y}_2)$ represent the incomes that would be received in any two states of the world in a given period, then (3.15) states the familiar result that *in each period*, the first-best transfer, $\tau^*(w, y_1, y_2)$, is the transfer for which each person's marginal rate of substitution of consumption across any two states will be equalized. This has the standard interpretation of the first-best risk-sharing outcome. Since effort choices are fully observable, the contract does not need to offer incentives for effort. The transfer function $\tau^*(w, y_1, y_2)$ is used purely to distribute the risks arising from income uncertainty. Furthermore if, say, the principal were risk neutral, i.e. u' constant and $u'' = 0$, then $\frac{u_{c_1}(y_1 + \tau^*(w, y_1, y_2))}{u_{c_1}(\tilde{y}_1 + \tau^*(w, \tilde{y}_1, \tilde{y}_2))} = 1$ and under condition (3.15) the agent would receive a constant level of consumption across states.

Evaluating (3.12) at (y_1, y_2) and $(\tilde{y}_1, \tilde{y}_2) \in [0, \bar{y}]^2$ implies

$$U'[V^*(w, y_1, y_2)] = U'[V^*(w, \tilde{y}_1, \tilde{y}_2)]. \quad (3.16)$$

Assuming $U''(w) \neq 0$, then $U'[V^*(w, y_1, y_2)] = U'[V^*(w, \tilde{y}_1, \tilde{y}_2)]$ if and only if $V^*(w, y_1, y_2) = V^*(w, \tilde{y}_1, \tilde{y}_2)$. Therefore (3.16) tells us that for any utility payoff $w \in \Lambda$ received by the agent at t , the promised future utility payoff at $t + 1$, given by $V^*(w, y_1, y_2)$, is the same irrespective of the incomes realised at t . So we establish that with $U(w)$ non-linear, $\frac{\partial V^*(w, y_1, y_2)}{\partial y_i} = 0$ ($i = 1, 2$). We can obtain this result more formally as follows. Under the assumption that $V^*(w, y_1, y_2)$ is differentiable, differentiate (3.12) with respect to y_1 and y_2 to get

$$U''(V^*(w, y_1, y_2)) V_{y_i}^* = 0, i = 1, 2. \quad (3.17)$$

Expression (3.17) confirms that under $U'' \neq 0$ we must have $V_{y_i}^* = 0$, for all i .

Hence the function $V^*(w, y_1, y_2)$ is such that the incomes realised today have no impact on promised future utilities. Both principal and agent receive the same utility payoffs (w and $U(w)$ respectively) each period. We may interpret this as follows. If effort choices were *not* observable, the function $V^*(w, y_1, y_2)$ could be used to exert an intertemporal control over effort by, say, punishing a low (rewarding a high) income today with a reduced (increased) utility payoff tomorrow. However since both effort levels are observable, each person chooses the optimal level of effort each period and so there is no need to control effort by varying the promised future utility payoffs to the agent and to the principal.

We now consider the effect of alternative income realizations, in any given period, on the efficient transfer $\tau^*(w, y_1, y_2)$ and on the optimal allocation of consumption between principal and agent.

Proposition 1. Let

$$c_1^* = c_1^*(w, y_1, y_2) = y_1 + \tau^*(w, y_1, y_2)$$

and

$$c_2^* = c_2^*(w, y_1, y_2) = y_2 - \tau^*(w, y_1, y_2)$$

represent the consumption levels accruing to the principal and the agent, respectively, under the first-best contract. Also let $\rho_1^* = -\frac{u_{c_1 c_1}(y_1 + \tau^*(w, y_1, y_2))}{u_{c_1}(y_1 + \tau^*(w, y_1, y_2))}$ and $\rho_2^* = -\frac{v_{c_2 c_2}(y_2 - \tau^*(w, y_1, y_2))}{v_{c_2}(y_2 - \tau^*(w, y_1, y_2))}$ represent the coefficients of absolute risk aversion, evaluated at the first-best transfer function, for the principal and the agent respectively. Then

$$\frac{\partial c_1^*}{\partial y_1} = \frac{\partial c_1^*}{\partial y_2} = \frac{\rho_2^*}{\rho_1^* + \rho_2^*} \in [0, 1]$$

and

$$\frac{\partial c_2^*}{\partial y_1} = \frac{\partial c_2^*}{\partial y_2} = \frac{\rho_1^*}{\rho_1^* + \rho_2^*} \in [0, 1].$$

Proof. See Appendix. ■

This result establishes that under the first-best contract (i) each individual's consumption is monotonically increasing in the incomes, and (ii) any increase in income is completely shared between the principal and agent according to their degree of risk-aversion.

3.5.2 Evolution of the Efficient Contract Over Time

In the preceding section we established that under $U''(\cdot) \neq 0$ we have $V_{y_i}^* = 0$ ($i = 1, 2$) and so the agent's promised future utility, $V^*(w, y_1, y_2)$, was the same irrespective of the incomes realized today. Our next result demonstrates that under the efficient contract, this promised future utility is the same as the utility received today.

Proposition 2. For each $w \in \Lambda$, $w = V^*(w, y_1, y_2)$ for all $(y_1, y_2) \in [0, \bar{y}]^2$.

Proof. See Appendix. ■

We interpret proposition 2 as follows. If w is the agent's expected discounted utility today, then the principal and agent choose effort levels $a_1^*(w)$ and $a_2^*(w)$, respectively, a transfer of $\tau^*(w, y_1, y_2)$ occurs today and the agent's promised expected discounted utility tomorrow, $V^*(w, y_1, y_2)$, is the same as today's. The contract tomorrow is therefore given by

$$\tau^*(V^*(w, y_1, y_2), y_1, y_2) = \tau^*(w, y_1, y_2),$$

$$V^*(V^*(w, y_1, y_2), y_1, y_2) = V^*(w, y_1, y_2)$$

and

$$a_i^*(V^*(w, y_1(w), y_2(w))) = a_i^*(w), i = 1, 2,$$

and is the same as the contract today, irrespective of the incomes realised today.

Under the efficient contract, therefore, $w_{t+1} = w_t, \forall t \in T$, and tomorrow's contract is always the same as the contract today. By contrast, under the repeated single agency

model of Spear and Srivastava (1987), there was only a single output level (the critical output) for which $w_{t+1} = w_t$. For all output realizations other than the critical output, the promised future utility was different from current utility and hence tomorrow's contract in general was different from the contract today. Under the repeated double agency problem analyzed later, we will show that there exists a single *pair* of income levels for which $w_{t+1} = w_t$ and for which the contract tomorrow will be the same as that offered today. For all other income pairs realised today, we show that tomorrow's utility payoff will differ from today's, and so tomorrow's contract will also differ from today's. In the repeated double agency case it is important to note that the rewards and punishments, in any given period, must respond not only to the income realised by the agent, but also to the income realised by the principal.

Inspection of condition (3.11) tells us that this expression takes the same form as the equilibrium condition for the efficient transfer in a one-period model of risk-sharing. The function $\tau^*(w, y_1, y_2)$ therefore exhibits the same qualitative features as the optimal transfer function from a static problem. The difference here is that in a repeated relationship, these features are required to hold each period. Our next result confirms this. It simply tells us that $\tau^*(w, y_1, y_2)$ is such that the marginal utility ratio is constant from one period to the next.

Proposition 3. Let y_{it-1} ($i = 1, 2$) be the incomes realised at $t - 1$ and τ_{t-1}^* the known transfer at $t - 1$. Then, for each $w \in \Lambda$,

$$\frac{u_{c_1}(y_{1t-1} + \tau_{t-1}^*)}{v_{c_2}(y_{2t-1} - \tau_{t-1}^*)} = \frac{u_{c_1}(y_1 + \tau^*(w, y_1, y_2))}{v_{c_2}(y_2 - \tau^*(w, y_1, y_2))}, \text{ for all } (y_1, y_2) \in [0, \bar{y}]^2. \quad (3.18)$$

Proof. See Appendix. ■

This result just tells us that the efficient transfer function acts to keep the marginal utility ratio constant from one period to the next. Furthermore, re-arranging (3.18) tells us that under the efficient contract, the marginal rate of substitution of consumption

between any two consecutive periods is equalized across individuals. We can interpret this result as saying that the efficient transfer function, in addition to allocating risk across states in any given period, will also take advantage of opportunities for intertemporal consumption smoothing.

3.6 The Second-Best (Constrained-Efficient) Contract

When both agents' effort choices are unobservable, any risk-sharing contract must also be incentive compatible. In this case the principal now solves the problem stated in (P). The second-best contract, which we denote by

$$\left\{ \hat{\tau}(w, y_1, y_2), \hat{V}(w, y_1, y_2), \hat{a}_1(w), \hat{a}_2(w) \right\},$$

must satisfy

$$\frac{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))} = - \frac{\left(\hat{\lambda}_1(w) + \hat{\lambda}_2(w) \frac{f_{2a_2}(y_2, \hat{a}_2(w))}{f_2(y_2, \hat{a}_2(w))} \right)}{\left(1 - \hat{\lambda}_3(w) \frac{f_{1a_1}(y_1, \hat{a}_1(w))}{f_1(y_1, \hat{a}_1(w))} \right)}, \quad (3.19)$$

$$U'(\hat{V}(w, y_1, y_2)) = \frac{\left(\hat{\lambda}_1(w) + \hat{\lambda}_2(w) \frac{f_{2a_2}(y_2, \hat{a}_2(w))}{f_2(y_2, \hat{a}_2(w))} \right)}{\left(1 - \hat{\lambda}_3(w) \frac{f_{1a_1}(y_1, \hat{a}_1(w))}{f_1(y_1, \hat{a}_1(w))} \right)}, \quad (3.20)$$

$$\frac{\partial Eu}{\partial a_1} - \hat{\lambda}_1(w) \frac{\partial Ev}{\partial a_1} - \hat{\lambda}_2(w) \frac{\partial}{\partial a_1} \left(\frac{\partial Ev}{\partial a_2} \right) - \hat{\lambda}_3(w) \frac{\partial^2 Eu}{\partial a_1^2} = 0 \quad (3.21)$$

and

$$\frac{\partial Eu}{\partial a_2} - \hat{\lambda}_1(w) \frac{\partial Ev}{\partial a_2} - \hat{\lambda}_2(w) \frac{\partial^2 Ev}{\partial a_2^2} - \hat{\lambda}_3(w) \frac{\partial}{\partial a_2} \left(\frac{\partial Eu}{\partial a_1} \right) = 0, \quad (3.22)$$

where $\hat{\lambda}_1(w)$ is the multiplier on the agent's utility promise constraint, $\hat{\lambda}_2(w)$ is the multiplier on the agent's incentive compatibility constraint, and $\hat{\lambda}_3(w)$ is the multiplier on the principal's incentive compatibility constraint. Note that both (3.21) and (3.22) are evaluated at the second-best contract. Also, in evaluating (3.19) and (3.20) we have taken

advantage of assumption 3 that individual incomes are independent random variables.

3.6.1 The Second-Best Contract Within a Given Time Period

For any $w \in \Lambda$, equations (3.19) - (3.22) provide conditions satisfied by

$$\left\{ \hat{\tau}(w, y_1, y_2), \hat{V}(w, y_1, y_2), \hat{a}_1(w), \hat{a}_2(w) \right\}$$

at any given time. Since the marginal utilities of consumption are positive, (3.19) and (3.20) imply that the constrained Pareto frontier is negatively sloped, i.e. $U' \left[\hat{V}(w, y_1, y_2) \right] < 0$. Therefore under the second-best contract we can raise one person's utility payoff only by reducing the utility payoff of the other. Throughout we assume that the constrained Pareto frontier is strictly concave, i.e. $U''(\cdot) < 0$.⁷

From the Envelope Theorem we can show that $U'(w) = \hat{\lambda}_1(w)$ and so it follows that $\hat{\lambda}_1(w) < 0$. Also we assume $\hat{\lambda}_2(w) < 0$ and $\hat{\lambda}_3(w) < 0$.⁸ We can now proceed with our characterization of the optimal contract under repeated double moral hazard. From our discussion in the previous section we know that unless the right hand side of (3.19) is constant, then $\hat{\tau}(w, y_1, y_2)$ cannot be efficient from a risk-sharing point of view. Our next result shows that $\hat{\tau}(w, y_1, y_2)$ cannot be the efficient transfer schedule.

Proposition 4. As long as we do not have both $\hat{\lambda}_2(w)$ and $\hat{\lambda}_3(w)$ being zero, then the right hand side of (3.19) is non-constant and the transfer schedule $\hat{\tau}(w, y_1, y_2)$ cannot be efficient from a risk-sharing point of view.

Proof. See Appendix. ■

An analogous result can easily be obtained for condition (3.20) and so, as long as we do not have both $\hat{\lambda}_2(w)$ and $\hat{\lambda}_3(w)$ being zero, then the second-best utility payoff

⁷Spear and Srivastava (1987) proved this formally for the repeated single moral hazard model and we expect this result to also hold for the repeated double moral hazard case.

⁸Because of the high level of generality of the model it is not possible to determine the signs of $\hat{\lambda}_2(w)$ and $\hat{\lambda}_3(w)$. However as we shall see, the implications for the behaviour of the optimal contract of assuming $\hat{\lambda}_2(w)$ and $\hat{\lambda}_3(w)$ to be strictly negative actually accord very well with our intuition.

schedule, $\hat{V}(w, y_1, y_2)$, must deviate from the efficient utility payoff schedule given by $V^*(w, y_1, y_2) = w$, for all $(y_1, y_2) \in [0, \bar{y}]^2$.

Consider any pair of income levels (y_1, y_2) and $(\tilde{y}_1, \tilde{y}_2) \in [0, \bar{y}]^2$. From proposition 4 it must be the case that

$$\frac{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))} \neq \frac{u_{c_1}(\tilde{y}_1 + \hat{\tau}(w, \tilde{y}_1, \tilde{y}_2))}{v_{c_2}(\tilde{y}_2 - \hat{\tau}(w, \tilde{y}_1, \tilde{y}_2))}. \quad (3.23)$$

Condition (3.23) says that as incomes vary across states of the world at any given time, the marginal utility ratio does not remain constant across states. Re-arranging (3.23) implies

$$\frac{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{u_{c_1}(\tilde{y}_1 + \hat{\tau}(w, \tilde{y}_1, \tilde{y}_2))} \neq \frac{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}{v_{c_2}(\tilde{y}_2 - \hat{\tau}(w, \tilde{y}_1, \tilde{y}_2))} \quad (3.24)$$

and so under the second-best transfer, $\hat{\tau}(w, y_1, y_2)$, the marginal rates of substitution of consumption across states will not be equalized across individuals. Even if the principal were risk neutral (i.e. u' constant and $u'' = 0$) so that $\frac{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{u_{c_1}(\tilde{y}_1 + \hat{\tau}(w, \tilde{y}_1, \tilde{y}_2))} = 1$, the agent would nevertheless not receive the same consumption in each state and would therefore be required to bear some risk. This outcome has the standard interpretation that when effort is unobservable the second-best contract does not provide an efficient level of risk-sharing. This is necessary in order to offer some incentives for effort. However the characterization in condition (3.19) tells us that the second-best transfer, $\hat{\tau}(w, y_1, y_2)$, must be chosen so as to achieve the correct balance between sharing risk and offering the right incentives to *both* individuals. We will return to this point later.

Evaluating (3.20) at (y_1, y_2) and $(\tilde{y}_1, \tilde{y}_2)$ implies

$$U'[\hat{V}(w, y_1, y_2)] \neq U'[\hat{V}(w, \tilde{y}_1, \tilde{y}_2)] \quad (3.25)$$

and hence

$$\hat{V}(w, y_1, y_2) \neq \hat{V}(w, \tilde{y}_1, \tilde{y}_2). \quad (3.26)$$

This establishes that, for any given w at time t , the promised utility payoffs at $t + 1$

depend upon the incomes realised at t . Equivalently, $\hat{V}_{y_i}(w, y_1, y_2) \neq 0, i = 1, 2$.⁹

These features of the second-best contract are in contrast to those of the efficient contract. Under the efficient contract perfect risk-sharing was achieved each period. Also under the efficient contract the promised future utility payoffs did not depend on the incomes that were currently realised. As a result, the efficient contract did not alter from one period to the next. In contrast under the second-best contract, perfect risk-sharing each period is not achieved. Furthermore the contract offered tomorrow will be influenced by the incomes realized today. Later on, in propositions 8 and 9, we provide more results on the relationship between the first-best (efficient) contract and our second-best (constrained-efficient) contract.

Comparing (3.19) and (3.20) with analogous conditions obtained for the problem of repeated *single* moral hazard (see Spear and Srivastava, 1987, conditions 4.3a and 4.4) we see that the characterization of the repeated double moral hazard contract must be different from the characterization of the optimal contract under repeated single moral hazard. As we now discuss, the key difference reflects the fact that under double moral hazard, the optimal contract must *simultaneously* take account of incentives for *both* individuals.

From proposition 4 it follows that as incomes vary across states, the *likelihood ratios* $\frac{f_{ia_i}(y_i, \hat{a}_i(w))}{f_i(y_i, \hat{a}_i(w))}$ ($i = 1, 2$) must also vary. Therefore, as long as we do not have both $\hat{\lambda}_2(w)$ and $\hat{\lambda}_3(w)$ being equal to zero, it follows that the right hand sides, and hence also the left hand sides, of (3.19) and (3.20) must vary under alternative income realizations. But this just tells us that under the second-best contract there is some deviation from the efficient level of risk-sharing each period, and from the efficient structure of rewards and penalties over the long term. However these deviations must be different in character from the deviations generated when there is only single moral hazard.

From conditions (3.19) and (3.20) we see that the optimal contract under repeated

⁹Note that condition (3.25) rules out linearity of the constrained Pareto frontier. If the slope of the frontier differs at different points in its domain, then the frontier cannot be linear.

double moral hazard depends on the distribution of y_i and on the functional relationship between this distribution and a_i , $i = 1, 2$. This contrasts with the optimal repeated single agency contract which depends only on the distribution of the agent's income and on the relationship between that distribution and the agent's effort. It also contrasts with the first-best contract which is wholly unrelated to the income distribution of either the principal or the agent, or to the relationship between these distributions and the effort levels.

Following Holmström (1979), the larger is $|f_{ia_i}|$ the stronger the incentive effect, for person i , of deviating from optimal risk-sharing. Also the greater is f_i the more costly, in terms of the risk-sharing benefits lost to person i , are such deviations. Thus $\frac{|f_{ia_i}|}{f_i}$ may be interpreted as a benefit-cost ratio, for person i , for deviation from optimal risk-sharing. Note that the likelihood ratio, $\frac{f_{ia_i}}{f_i}$, is just the derivative of the maximum likelihood function, $\log f_i$, when a_i is viewed as an unknown parameter. Therefore $\frac{|f_{ia_i}|}{f_i}$ measures how strongly one would wish to infer, based on the observation y_i , that person i did not take the assumed action. Conditions (3.19) and (3.20) therefore say that the optimal level of risk-sharing under repeated double moral hazard must take account of such inferences for *both* individuals, and that any punishments or rewards (expressed in terms of deviations from first-best risk-sharing) should be proportional to the likelihood ratios of both the principal and the agent. Later on, in propositions 8 and 9, we will say more about the precise nature of these deviations. We now proceed to offer some characterizations of the second-best contract. However before doing so we will first need the technical results in Lemmas 1 and 2.

Lemma 1. For any $a_i(w)$ ($i = 1, 2$), under the *monotone likelihood ratio condition* (*mlrc*) there exists a unique income level $\tilde{y}_i(w)$ such that $f_{ia_i}(\tilde{y}_i(w), a_i(w)) = 0$. For $y_i < \tilde{y}_i(w)$ we have $f_{ia_i}(y_i(w), a_i(w)) < 0$. For $y_i > \tilde{y}_i(w)$ we have $f_{ia_i}(y_i(w), a_i(w)) > 0$.

Proof. See Appendix. ■

Lemma 1 implies that under *mlrc*, more effort by person i will reduce the likelihood of receiving incomes below $\tilde{y}_i(w)$ and will increase the likelihood of receiving incomes

above $\tilde{y}_i(w)$.

Lemma 2. For all $w \in \Lambda$ and $(y_1, y_2) \in [0, \bar{y}]^2$,

$$\left(1 - \hat{\lambda}_3(w) \frac{f_{1a_1}(y_1, \hat{a}_1(w))}{f_1(y_1, \hat{a}_1(w))}\right) > 0$$

and

$$\left(\hat{\lambda}_1(w) + \hat{\lambda}_2(w) \frac{f_{2a_2}(y_2, \hat{a}_2(w))}{f_2(y_2, \hat{a}_2(w))}\right) < 0.$$

Proof. See Appendix. ■

In the remainder of this chapter we examine how the second-best contract allocates risk and provides incentives. In the rest of this section we examine the second-best transfer and the second-best allocation of consumption under alternative income realizations. In the next section we look at certain aspects of the evolution of the contract from one period to the next. Throughout we draw comparisons between the double moral hazard contract, on the one hand, and the full-information and single moral hazard contracts, on the other. We can now state the following result.

Proposition 5. Under the second-best contract, increases in the level of y_1 realized today lead to a reduction in today's marginal utility ratio, $\frac{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}$, while increases in the level of y_2 realized today lead to an increase in this marginal utility ratio. Let

$$\hat{c}_1 = \hat{c}_1(w, y_1, y_2) = y_1 + \hat{\tau}(w, y_1, y_2)$$

and

$$\hat{c}_2 = \hat{c}_2(w, y_1, y_2) = y_2 - \hat{\tau}(w, y_1, y_2)$$

represent the current levels of consumption that accrue to the principal and the agent, respectively, under the second-best contract. Then under *mlrc*, $\hat{\lambda}_2(w) < 0$ and $\hat{\lambda}_3(w) < 0$, we have $\frac{\partial \hat{c}_1}{\partial y_1} > 0$ and $\frac{\partial \hat{c}_2}{\partial y_2} > 0$. Also, the effect of changes in y_1 or y_2 on the transfer

$\hat{\tau}(w, y_1, y_2)$ is ambiguous.

Proof. See Appendix. ■

This result tells us that although the effect of different income realizations on the size of the second-best transfer between principal and agent is ambiguous, the effect on absolute consumption levels is quite clear. *Ceteris paribus*, a better income realization by person i at time t is rewarded immediately by a rise in i 's time t consumption level, $i = 1, 2$. We also see that a better income realization by person i at time t leads to a fall in person i 's marginal utility relative to person j 's marginal utility ($i \neq j$). This suggests that greater income for person i is rewarded not only by a rise in i 's absolute consumption but also by a 'somewhat higher' consumption for person i relative to the consumption of person j .

3.6.2 Evolution of the Second-Best Contract Over Time

In the repeated single agency model analyzed by Spear and Srivastava (1987) it was shown that each period there was a single output level which, if realised, meant that next period's utility payoff (and hence next period's contract) would be the same as this period's utility payoff and contract. Our next result demonstrates that this feature of the optimal single agency contract generalizes under the double agency problem analyzed here.

Proposition 6. Let $(\tilde{y}_1(w), \tilde{y}_2(w)) \in [0, \bar{y}]^2$ represent the current income levels for which $f_{1a_1}(\tilde{y}_1(w), \hat{a}_1(w)) = 0$ and $f_{2a_2}(\tilde{y}_2(w), \hat{a}_2(w)) = 0$. (From lemma 1 we established that such income levels existed and were unique). Then for every $w \in \Lambda$,

$$w = \hat{V}(w, \tilde{y}_1(w), \tilde{y}_2(w)) \quad (3.27)$$

and

$$\frac{u_{c_1}(\tilde{y}_1(w) + \hat{\tau}(w, \tilde{y}_1(w), \tilde{y}_2(w)))}{v_{c_2}(\tilde{y}_2(w) - \hat{\tau}(w, \tilde{y}_1(w), \tilde{y}_2(w)))} = -\hat{\lambda}_1(w). \quad (3.28)$$

Proof. See Appendix. ■

We may interpret this result as follows. If w is the agent's expected discounted utility today and incomes $(\tilde{y}_1(w), \tilde{y}_2(w))$ are realised today, then $\hat{a}_1(w)$ and $\hat{a}_2(w)$ are chosen today, a transfer of $\hat{\tau}(w, \tilde{y}_1(w), \tilde{y}_2(w))$ occurs today, and the agent's promised expected discounted utility tomorrow is given by $\hat{V}(w, \tilde{y}_1(w), \tilde{y}_2(w))$ and is the same as his utility payoff today. The fact that $w = \hat{V}(w, \tilde{y}_1(w), \tilde{y}_2(w))$ implies that whenever incomes $(\tilde{y}_1(w), \tilde{y}_2(w))$ are realised, the contract tomorrow is the same as the contract today. Note that the utility payoff, $\hat{V}(w, \tilde{y}_1(w), \tilde{y}_2(w))$, can be thought of as the 'first-best' utility payoff. Also the transfer, $\hat{\tau}(w, \tilde{y}_1(w), \tilde{y}_2(w))$, can be thought of as the 'first-best' transfer, when $-\hat{\lambda}_1(w)$ represents the weight given to the agent's preferences in solving the full-information first-best problem. Note that $\hat{\tau}(w, \tilde{y}_1(w), \tilde{y}_2(w))$ is chosen to satisfy $\frac{u_{c_1}(\tilde{y}_1(w) + \hat{\tau}(w, \tilde{y}_1(w), \tilde{y}_2(w)))}{v_{c_2}(\tilde{y}_2(w) - \hat{\tau}(w, \tilde{y}_1(w), \tilde{y}_2(w)))} = -\hat{\lambda}_1(w)$. Finally, the marginal utility ratio, $\frac{u_{c_1}(\tilde{y}_1(w) + \hat{\tau}(w, \tilde{y}_1(w), \tilde{y}_2(w)))}{v_{c_2}(\tilde{y}_2(w) - \hat{\tau}(w, \tilde{y}_1(w), \tilde{y}_2(w)))}$, can similarly be thought of as the 'first-best' marginal utility ratio.

The next result examines the future utility payoff when incomes $(y_1(w), y_2(w)) \neq (\tilde{y}_1(w), \tilde{y}_2(w))$ are realised today.

Proposition 7. Under *mlrc*, $U''(w) < 0$, $\hat{\lambda}_2(w) < 0$, and $\hat{\lambda}_3(w) < 0$ we have

$$\hat{V}_{y_1}(w, y_1, y_2) < 0$$

and

$$\hat{V}_{y_2}(w, y_1, y_2) > 0.$$

Proof. See Appendix. ■

This result demonstrates that the schedule of future utilities is monotonic in individ-

uals' current incomes. In particular, the agent's future utility payoff is monotonically increasing in the agent's own income and monotonically decreasing in the principal's income. Also since $U'(w) < 0$, it follows that the principal's future utility payoff is also monotonically increasing in the principal's own income and monotonically decreasing in the agent's income. Proposition 7 also implies that increases in *both* y_1 and y_2 will have offsetting effects on tomorrow's utility promises.

We can now pull together the results of propositions 5, 6 and 7 in order to say something about the relationship between the second-best and the first-best transfer, and between the second-best and the first-best utility payoff.

Proposition 8. (i) If $y_1 < \tilde{y}_1(w)$ and $y_2 = \tilde{y}_2(w)$, or $y_1 \leq \tilde{y}_1(w)$ and $y_2 > \tilde{y}_2(w)$, then $\hat{V}(w, y_1, y_2) > w$ and $\frac{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))} > \frac{u_{c_1}(\tilde{y}_1(w) + \hat{\tau}(w, \tilde{y}_1(w), \tilde{y}_2(w)))}{v_{c_2}(\tilde{y}_2(w) - \hat{\tau}(w, \tilde{y}_1(w), \tilde{y}_2(w)))}$.
(ii) If $y_1 > \tilde{y}_1(w)$ and $y_2 \leq \tilde{y}_2(w)$, or $y_1 = \tilde{y}_1(w)$ and $y_2 < \tilde{y}_2(w)$, then $\hat{V}(w, y_1, y_2) < w$ and $\frac{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))} < \frac{u_{c_1}(\tilde{y}_1(w) + \hat{\tau}(w, \tilde{y}_1(w), \tilde{y}_2(w)))}{v_{c_2}(\tilde{y}_2(w) - \hat{\tau}(w, \tilde{y}_1(w), \tilde{y}_2(w)))}$.
(iii) If $y_1 > \tilde{y}_1(w)$ and $y_2 > \tilde{y}_2(w)$, or $y_1 < \tilde{y}_1(w)$ and $y_2 < \tilde{y}_2(w)$, then the effect on $\hat{V}(w, y_1, y_2)$ *vis-a-vis* w and on $\frac{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}$ *vis-a-vis* $\frac{u_{c_1}(\tilde{y}_1(w) + \hat{\tau}(w, \tilde{y}_1(w), \tilde{y}_2(w)))}{v_{c_2}(\tilde{y}_2(w) - \hat{\tau}(w, \tilde{y}_1(w), \tilde{y}_2(w)))}$ is ambiguous.

Proof. These results follow directly by combining the results of propositions 5, 6 and 7. ■

Proposition 8(i) can be interpreted as saying that whenever the principal receives a 'relatively low income' and/or the agent a 'relatively high one' (i.e. relative to their critical income), then the agent is rewarded (principal is punished) tomorrow with a higher (lower) utility payoff than the payoff he received today (i.e. than the first-best payoff). Also under this outcome the current transfer will be chosen so that the current ratio of the principal's to the agent's marginal utility is high relative to the first-best marginal utility ratio. On the other hand, proposition 8(ii) can be interpreted as saying that whenever the principal receives a 'relatively high income' and/or the agent a 'relatively low one', then the agent is punished (principal is rewarded) tomorrow with a lower (higher) utility payoff than the payoff he received today. Also the current transfer will

be chosen so that the current ratio of the principal's to the agent's marginal utility is low relative to the first-best marginal utility ratio. Finally proposition 8(iii) tells us that if *both* principal and agent receive a 'relatively high' income (or if they *both* receive a 'relatively low income'), then the effect this has on the current principal-agent marginal utility ratio *vis-a-vis* the first-best ratio, and on the future utility payoffs *vis-a-vis* the current payoffs, is ambiguous and cannot be established without further information on preferences and income distributions.

The implication of proposition 8(iii) is that the second-best contract, in terms of the structure of rewards and penalties offered within any given period, is less sensitive to the performance of any single individual than would have been the case under a repeated single agency problem. Recall that under repeated single agency a good performance by the agent is *always* rewarded with an increase in today's consumption *and* an increase in tomorrow's utility. Likewise a bad performance on the part of the agent will *always* be punished with a reduction in today's consumption *and* a reduction in tomorrow's utility payoff. In the single agency setting these rewards and punishments are entirely unaffected by the performance of another individual (say the principal). In the current setting, however, we see that the rewards and punishments offered to one person will depend not only on that person's performance but also on the performance of the other person.

The next result discusses the evolution of the second-best marginal utility ratio and of the second-best utility promises over time.

Proposition 9. Let (\bar{y}_1, \bar{y}_2) represent the known incomes realized at time t . Let $w' = \hat{V}(w, \bar{y}_1, \bar{y}_2)$ represent the agent's utility payoff at $t + 1$, (y'_1, y'_2) the incomes realized at $t + 1$, and

$$(\tilde{y}'_1, \tilde{y}'_2) = (\tilde{y}_1(w'), \tilde{y}_2(w')) \in [0, \bar{y}]^2$$

the $t + 1$ critical income levels defined by

$$f_{1a_1}(\tilde{y}_1(w'), \hat{a}_1(w')) = 0$$

and

$$f_{2a_2}(\tilde{y}_2(w'), \hat{a}_2(w')) = 0.$$

Finally, let $\frac{u_{c_1}(y'_1 + \hat{\tau}(w', y'_1, y'_2))}{v_{c_2}(y'_2 - \hat{\tau}(w', y'_1, y'_2))}$ represent the ratio of the principal's to the agent's marginal utility at $t + 1$, evaluated at the optimal $t + 1$ transfer, $\hat{\tau}(w', y'_1, y'_2)$, for given $t + 1$ incomes. Then, under *mlrc*, $\hat{\lambda}_2(w') < 0$, $\hat{\lambda}_3(w') < 0$, and taking account of lemma 2 we have the following:

- (i) If $(y'_1, y'_2) = (\tilde{y}'_1, \tilde{y}'_2)$, then $\frac{u_{c_1}(y'_1 + \hat{\tau}(w', y'_1, y'_2))}{v_{c_2}(y'_2 - \hat{\tau}(w', y'_1, y'_2))} = \frac{u_{c_1}(\bar{y}_1 + \hat{\tau}(w, \bar{y}_1, \bar{y}_2))}{v_{c_2}(\bar{y}_2 - \hat{\tau}(w, \bar{y}_1, \bar{y}_2))}$ and $\hat{V}(w', y'_1, y'_2) = \hat{V}(w, \bar{y}_1, \bar{y}_2)$.
- (ii) If $y'_1 < \tilde{y}'_1$ and $y'_2 = \tilde{y}'_2$, or $y'_1 \leq \tilde{y}'_1$ and $y'_2 > \tilde{y}'_2$, then $\frac{u_{c_1}(y'_1 + \hat{\tau}(w', y'_1, y'_2))}{v_{c_2}(y'_2 - \hat{\tau}(w', y'_1, y'_2))} > \frac{u_{c_1}(\bar{y}_1 + \hat{\tau}(w, \bar{y}_1, \bar{y}_2))}{v_{c_2}(\bar{y}_2 - \hat{\tau}(w, \bar{y}_1, \bar{y}_2))}$ and $\hat{V}(w', y'_1, y'_2) > \hat{V}(w, \bar{y}_1, \bar{y}_2)$.
- (iii) If $y'_1 > \tilde{y}'_1$ and $y'_2 \leq \tilde{y}'_2$, or $y'_1 = \tilde{y}'_1$ and $y'_2 < \tilde{y}'_2$, then $\frac{u_{c_1}(y'_1 + \hat{\tau}(w', y'_1, y'_2))}{v_{c_2}(y'_2 - \hat{\tau}(w', y'_1, y'_2))} < \frac{u_{c_1}(\bar{y}_1 + \hat{\tau}(w, \bar{y}_1, \bar{y}_2))}{v_{c_2}(\bar{y}_2 - \hat{\tau}(w, \bar{y}_1, \bar{y}_2))}$ and $\hat{V}(w', y'_1, y'_2) < \hat{V}(w, \bar{y}_1, \bar{y}_2)$.
- (iv) If $y'_1 > \tilde{y}'_1$ and $y'_2 > \tilde{y}'_2$, or $y'_1 < \tilde{y}'_1$ and $y'_2 < \tilde{y}'_2$, then the effect on $\frac{u_{c_1}(y'_1 + \hat{\tau}(w', y'_1, y'_2))}{v_{c_2}(y'_2 - \hat{\tau}(w', y'_1, y'_2))}$ *vis-a-vis* $\frac{u_{c_1}(\bar{y}_1 + \hat{\tau}(w, \bar{y}_1, \bar{y}_2))}{v_{c_2}(\bar{y}_2 - \hat{\tau}(w, \bar{y}_1, \bar{y}_2))}$ and on $\hat{V}(w', y'_1, y'_2)$ *vis-a-vis* $\hat{V}(w, \bar{y}_1, \bar{y}_2)$ is ambiguous.

Proof. See Appendix. ■

This result describes the relationship between the time $t + 1$ and the time t marginal utility ratios, and between the time $t + 2$ and the time $t + 1$ utility payoffs. Proposition 9(i) tells us that if, at $t + 1$, both agents obtain their $t + 1$ critical incomes, then the $t + 1$ marginal utility ratio will be the same as the known marginal utility ratio at t , and the $t + 2$ utility payoffs will be the same as the $t + 1$ utility payoffs. Proposition 9(ii) can be interpreted as saying that if, at time $t + 1$, the principal receives a ‘relatively low income’ and/or the agent a ‘relatively high one’ (this time relative to the $t + 1$ critical incomes), then the $t + 1$ principal-agent marginal utility ratio will be greater than the known principal-agent marginal utility ratio at t , and the agent’s (principal’s) $t + 2$ utility payoff will also be greater (lower) than the agent’s (principal’s) $t + 1$ utility payoff. Proposition 9(iii) can be similarly interpreted as saying that if, at time $t + 1$, the principal receives a ‘relatively high income’ and/or the agent a ‘relatively low one’, then

the $t + 1$ principal-agent marginal utility ratio will be lower than the known principal-agent marginal utility ratio at t , and the agent's (principal's) $t + 2$ utility payoff will also be lower (higher) than the agent's (principal's) $t + 1$ utility payoff. Finally, proposition 9(iv) just tells us that if, at time $t + 1$, *both* principal and agent perform well (or if they both perform badly), then we cannot predict the effect on the $t + 1$ marginal utility ratio *vis-a-vis* the time t ratio, nor on the $t + 2$ utility payoffs *vis-a-vis* the $t + 1$ payoffs. Again we see that the second-best contract, this time in terms of the structure of rewards and penalties offered over time, continues to be less sensitive to the performance of particular individuals than is the case under repeated single agency.

In order to offer a full characterization of the behaviour of the second-best contract, it is necessary to know how the critical incomes $(\tilde{y}_1(w), \tilde{y}_2(w))$ evolve over time, i.e. how they respond as w changes from one period to the next. The current critical income, $\tilde{y}_i(w)$, offers a benchmark level of income against which person i 's current performance can be judged. Suppose i ' current income is y_i . Then $y_i > \tilde{y}_i$ can be interpreted as saying that person i has performed 'relatively well' in the current period, while $y_i < \tilde{y}_i$ implies that person i 's current performance has been 'relatively poor'. Also, if \tilde{y}_i rises (falls) from one period to the next, then in the subsequent period it becomes more difficult (easier) for person i to do well than it had been in the first period.

We are therefore interested in knowing the sign of $\tilde{y}'_i(w)$, $i = 1, 2$. Remember that the critical incomes $\tilde{y}_1(w)$ and $\tilde{y}_2(w)$ were the income levels that satisfied

$$f_{1a_1}(\tilde{y}_1(w), \hat{a}_1(w)) = 0$$

and

$$f_{2a_2}(\tilde{y}_2(w), \hat{a}_2(w)) = 0,$$

respectively. Therefore the probability density function $f_i(y_i(w), a_i(w))$ implicitly defines a relationship between i 's critical income, $\tilde{y}_i(w)$, and i 's chosen level of effort,

$\hat{a}_i(w)$. For example, under the exponential density function given by

$$f_i(y_i, a_i) = \frac{1}{a_i} \exp \left\{ -\frac{y_i}{a_i} \right\}, i = 1, 2,$$

it is easy to show that $\tilde{y}_i(w) = \hat{a}_i(w)$ ¹⁰ and therefore $\text{sign } \tilde{y}'_i(w) = \text{sign } \hat{a}'_i(w)$, $i = 1, 2$. Knowing the sign of $\tilde{y}'_i(w)$, therefore, requires (a) choosing a particular probability density function, and (b) knowing the comparative statics of each individual's effort level as w changes. In the repeated single moral hazard problem analyzed by Spear and Srivastava (1987) the authors show that it is not possible to unambiguously sign the derivative of the agent's effort with respect to his utility payoff. Instead they identify conditions under which this derivative might be positive and conditions under which it might be negative. In some cases this sign is indeterminate. Because of the high level of generality of our repeated double agency model, it is unlikely that we will be able to obtain simple conditions under which the $\hat{a}'_i(w)$, $i = 1, 2$, are either negative or positive, and so we do not attempt to do so here.

We now turn to the final result of this chapter. Recall from proposition 3 that under the first-best contract

$$\frac{u_{c_1}(y_{1t-1} + \tau_{t-1}^*)}{v_{c_2}(y_{2t-1} - \tau_{t-1}^*)} = \frac{u_{c_1}(y_1 + \tau^*(w, y_1, y_2))}{v_{c_2}(y_2 - \tau^*(w, y_1, y_2))}, \text{ all } w \in \Lambda \text{ and } (y_1, y_2) \in [0, \bar{y}]^2. \quad (3.29)$$

Under the first-best contract, the marginal utility ratio is held constant from one period to the next. If the principal were risk neutral this would imply that the agent received a constant consumption stream over time, and so under the first-best contract the agent would obtain perfect consumption smoothing. However when *both* individuals are risk averse we can interpret (3.29) as meaning that the first-best contract tries to achieve the optimal level of consumption-smoothing over time, taking account of the risk-aversion of both individuals.

¹⁰Differentiate f_i with respect to a_i and evaluate the result at the particular effort level $\hat{a}_i(w)$. The critical income level \tilde{y}_i is the income level for which this expression is zero.

A standard result of the repeated single moral hazard model, obtained by Lambert (1983), Rogerson (1985a) and Spear and Srivastava (1987), is that the optimal contract must be such that the *expected* marginal utility ratio at any time t must always equal the known marginal utility ratio in the previous period. Suppose $\check{\tau}(w, y_1, y_2)$ represents the optimal transfer schedule under repeated single agency. Then, based on the framework and notation developed in this chapter, if only agent v (the ‘agent’) has an incentive problem we must have

$$E \left\{ \frac{u_{c_1}(y_1 + \check{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \check{\tau}(w, y_1, y_2))} \right\} = \frac{u_{c_1}(y_{1t-1} + \check{\tau}_{t-1})}{v_{c_2}(y_{2t-1} - \check{\tau}_{t-1})} > \frac{1}{E \left\{ \frac{1}{\frac{u_{c_1}(y_1 + \check{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \check{\tau}(w, y_1, y_2))}} \right\}}, \text{ all } w \in \Lambda, \quad (3.30)$$

where y_{it-1} ($i = 1, 2$) are the incomes realised at $t-1$, $\check{\tau}_{t-1}$ is the known transfer at $t-1$ and $E\{\cdot\}$ is the expectations operator over incomes (y_1, y_2) at t . Likewise (and as we demonstrate later in the proof of proposition 10) if only agent u (the ‘principal’) has an incentive problem then we must have

$$E \left\{ \frac{v_{c_2}(y_2 - \check{\tau}(w, y_1, y_2))}{u_{c_1}(y_1 + \check{\tau}(w, y_1, y_2))} \right\} = \frac{v_{c_2}(y_{2t-1} - \check{\tau}_{t-1})}{u_{c_1}(y_{1t-1} + \check{\tau}_{t-1})} > \frac{1}{E \left\{ \frac{1}{\frac{v_{c_2}(y_2 - \check{\tau}(w, y_1, y_2))}{u_{c_1}(y_1 + \check{\tau}(w, y_1, y_2))}} \right\}}, \text{ all } w \in \Lambda. \quad (3.31)$$

The inequalities in expressions (3.30) and (3.31) can be derived from Jensen’s inequality¹¹ whereas the equalities in these expressions represent the standard result of the repeated single moral hazard literature. Lambert (1983) interpreted the equality in

¹¹According to Jensen’s inequality if $h(\cdot)$ is a convex function and $E(X) < \infty$, then $h(E(X)) \leq E(h(X))$. In this context let $X = \frac{v_{c_2}}{u_{c_1}}$ and $h(X) = \frac{1}{X} = \frac{u_{c_1}}{v_{c_2}}$. Then noting that $\frac{1}{X}$ is strictly convex we have

$$\frac{1}{E \left\{ \frac{v_{c_2}}{u_{c_1}} \right\}} < E \left\{ \frac{u_{c_1}}{v_{c_2}} \right\}.$$

By a similar argument we also have

$$\frac{1}{E \left\{ \frac{u_{c_1}}{v_{c_2}} \right\}} < E \left\{ \frac{v_{c_2}}{u_{c_1}} \right\}.$$

(3.30), in the context of a two-period principal-agent relationship with a possibly risk neutral principal and a risk averse agent, as meaning that the optimal long-term contract “...smooths the agent’s income over time” (Lambert, 1983, p. 449). He also suggested that “...the principal spreads the risk of the first period outcome over as many periods as possible. By using the sharing rules to smooth the agent’s income, the principal can give the agent a lower expected payment in each period and still meet the agent’s minimum utility constraint” (Lambert, 1983, p. 449). Rogerson (1985a) established that the equality in (3.30) would apply to any two adjacent periods of a longer principal-agent relationship. He argued that “The repetition of a moral hazard relationship creates the opportunity for intertemporal risk sharing. The optimal contract always takes advantage of this, i.e., memory plays a role in the optimal contract. However, *because of the incentive problem the agent is not fully insured* (our emphasis) and the agent is left with a residual desire to intertemporally self-insure through the use of credit markets” (Rogerson, 1985a, p. 70).

We now present our final result. In doing so note that (3.30) can be equivalently expressed as

$$E \left\{ \frac{v_{c_2}(y_2 - \check{\tau}(w, y_1, y_2))}{u_{c_1}(y_1 + \check{\tau}(w, y_1, y_2))} \right\} > \frac{v_{c_2}(y_{2t-1} - \check{\tau}_{t-1})}{u_{c_1}(y_{1t-1} + \check{\tau}_{t-1})} = \frac{1}{E \left\{ \frac{1}{\frac{v_{c_2}(y_2 - \check{\tau}(w, y_1, y_2))}{u_{c_1}(y_1 + \check{\tau}(w, y_1, y_2))}} \right\}}, \text{ all } w \in \Lambda. \quad (3.32)$$

Proposition 10. Let (y_{1t-1}, y_{2t-1}) be the incomes realised at $t-1$ and $\hat{\tau}_{t-1}$ the known transfer at $t-1$. Then under repeated double agency (i.e. $\hat{\lambda}_2(w) \neq 0$ and $\hat{\lambda}_3(w) \neq 0$) we must have

$$\frac{1}{E \left\{ \frac{1}{\frac{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}} \right\}} < \frac{v_{c_2}(y_{2t-1} - \hat{\tau}_{t-1})}{u_{c_1}(y_{1t-1} + \hat{\tau}_{t-1})} < E \left\{ \frac{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))} \right\}, \text{ for all } w \in \Lambda, \quad (3.33)$$

where expectations are over incomes (y_1, y_2) at time t . If $\hat{\lambda}_3(w) = 0$ and $\hat{\lambda}_2(w) \neq 0$ then condition (3.32) holds. If $\hat{\lambda}_2(w) = 0$ and $\hat{\lambda}_3(w) \neq 0$ then condition (3.31) holds.

Proof. See Appendix. ■

According to proposition 10 condition (3.33) characterizes the optimal contract under repeated double moral hazard. It is easy to see that this expression represents a natural generalization of the two single moral hazard possibilities that are embodied, respectively, in conditions (3.32) and (3.31). Proposition 10 establishes that if only the agent (agent v) has a binding incentive constraint then (3.32) must be true, whereas if only the principal (agent u) has a binding incentive constraint then (3.31) must be true. If however *both* agents have binding incentive constraints then the optimal contract is characterized by (3.33) and we have an optimum that lies somewhere between the two single moral hazard extremes.

Proposition 10 illustrates that when the single agency setting is generalized to take account of the double incentive problem, the creation of an *additional* incentive problem implies that under the optimal contract there must be some deviation from the level of intertemporal consumption smoothing that would have been optimal under repeated single agency. There are two points to note about this deviation. First, the extent of deviation must be proportional to the benefit-cost ratio of doing so. (To see that this must be the case consider expressions (3.65) and (3.70) in the Appendix and recall our earlier interpretation of $\frac{|f_{ia_i}|}{f_i}$ as reflecting the benefit-cost ratio, associated with person i , of deviation from optimal risk-sharing). Second, inspection of condition (3.33) shows that when *both* agents have binding incentive constraints then

$$\frac{v_{c_2}(y_{2t-1} - \hat{\tau}_{t-1})}{u_{c_1}(y_{1t-1} + \hat{\tau}_{t-1})} < E \left\{ \frac{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))} \right\}$$

and

$$\frac{u_{c_1}(y_{1t-1} + \hat{\tau}_{t-1})}{v_{c_2}(y_{2t-1} - \hat{\tau}_{t-1})} < E \left\{ \frac{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))} \right\}.$$

This contrasts with the two single moral hazard cases under which

$$\frac{u_{c_1}(y_{1t-1} + \hat{\tau}_{t-1})}{v_{c_2}(y_{2t-1} - \hat{\tau}_{t-1})} = E \left\{ \frac{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))} \right\}$$

and

$$\frac{v_{c_2}(y_{2t-1} - \hat{\tau}_{t-1})}{u_{c_1}(y_{1t-1} + \hat{\tau}_{t-1})} < E \left\{ \frac{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))} \right\}$$

when only agent v has an incentive problem, or

$$\frac{v_{c_2}(y_{2t-1} - \hat{\tau}_{t-1})}{u_{c_1}(y_{1t-1} + \hat{\tau}_{t-1})} = E \left\{ \frac{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))} \right\}$$

and

$$\frac{u_{c_1}(y_{1t-1} + \hat{\tau}_{t-1})}{v_{c_2}(y_{2t-1} - \hat{\tau}_{t-1})} < E \left\{ \frac{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))} \right\}$$

when only agent u has an incentive problem.

3.7 Discussion and Areas for Further Work

Based on the analysis of this chapter we identify several lines of further inquiry. The first is an analysis of the effects of the double incentive problem on the effort choices of the principal and agent. It is a well known result that one of the inefficiencies of single moral hazard is that the agent chooses a lower level of effort than would have been chosen under full-information. However what effect does double moral hazard have on the effort choices of *both* principal and agent? Also how will individual effort levels evolve over time? Because of the high level of generality in our model we were unable to determine the sign of $\hat{a}'_i(w)$, $i = 1, 2$. Nevertheless it may be possible to establish the sign of the $\hat{a}'_i(w)$ under specific assumptions about the shape of individual preferences and about individuals' income distributions.

Second, concerns the role of binding contracts in influencing the properties of the optimal long-term contract. We can think of several circumstances under which binding

and enforceable contracts are unavailable and risk-sharing can be sustained only through informal agreements. Development of a tractable analysis of this question, while retaining the double agency problem, would be challenging and would represent a further departure from existing models. A third line of inquiry could be an empirical investigation into whether actual, long-term risk-sharing arrangements, in the double moral hazard context, possess all or some of the properties suggested by our analysis.

APPENDIX

Derivation of (3.6) and (3.7). For any history y^t and for $s = 1, 2, \dots$, note that since individual incomes are independently distributed from one period to the next, then $F(y^{t+s}; y^t, a_1, a_2)$ must be equivalent to $F(y_{t+1}, \dots, y_{t+s}; a_1, a_2)$. Then from the properties of joint density functions we have

$$\frac{d^{2s} F(y^{t+s}; y^t, a_1, a_2)}{dy_{2t+s} dy_{1t+s} \dots dy_{2t+1} dy_{1t+1}} = \frac{d^{2s} F(y_{t+1}, \dots, y_{t+s}; a_1, a_2)}{dy_{2t+s} dy_{1t+s} \dots dy_{2t+1} dy_{1t+1}} = f(y_{t+1}, \dots, y_{t+s}; a_1, a_2). \quad (3.34)$$

Again since incomes are independently distributed over time, and taking account of the fact that the probability density over incomes at any time t is contingent only on the effort levels chosen at t , we have

$$f(y_{t+1}, \dots, y_{t+s}; a_1, a_2) = \prod_{j=1}^s f(y_{1t+j}, y_{2t+j}, a_{1t+j}(y^{t+j-1}), a_{2t+j}(y^{t+j-1})). \quad (3.35)$$

Combining (3.34) and (3.35) implies that the distribution $F(y^{t+s}; y^t, a_1, a_2)$ can be expressed as

$$d^{2s} F(y^{t+s}; y^t, a_1, a_2) = \prod_{j=1}^s \{f(y_{1t+j}, y_{2t+j}, a_{1t+j}(y^{t+j-1}), a_{2t+j}(y^{t+j-1})) dy_{1t+j} dy_{2t+j}\}. \quad (3.36)$$

Now note that

$$U(y^t, \tau, a_1, a_2)$$

$$\begin{aligned}
&= \sum_{s=1}^{\infty} \beta^{s-1} \int \dots \int [u(y_{1t+s} + \tau_{t+s}(y^{t+s})) - a_{1t+s}(y^{t+s-1})] d^{2s} F(y^{t+s}; y^t, a_1, a_2) \\
&= \int \int [u(y_{1t+1} + \tau_{t+1}(y^{t+1})) - a_{1t+1}(y^t)] d^2 F(y^{t+1}; y^t, a_1, a_2) \\
&\quad + \sum_{s=2}^{\infty} \beta^{s-1} \int \dots \int [u(y_{1t+s} + \tau_{t+s}(y^{t+s})) - a_{1t+s}(y^{t+s-1})] d^{2s} F(y^{t+s}; y^t, a_1, a_2)
\end{aligned} \tag{3.37}$$

and so it follows that

$$\begin{aligned}
&U(y^{t+1}, \tau, a_1, a_2) \\
&= \sum_{s=1}^{\infty} \beta^{s-1} \int \dots \int [u(y_{1t+1+s} + \tau_{t+1+s}(y^{t+1+s})) - a_{1t+1+s}(y^{t+s})] d^{2s} F(y^{t+1+s}; y^{t+1}, a_1, a_2).
\end{aligned} \tag{3.38}$$

Substituting from (3.36) into (3.37) gives us

$$\begin{aligned}
&U(y^t, \tau, a_1, a_2) \\
&= \sum_{s=1}^{\infty} \beta^{s-1} \int \dots \int [u(y_{1t+s} + \tau_{t+s}(y^{t+s})) - a_{1t+s}(y^{t+s-1})] \\
&\quad \times \prod_{j=1}^s \{f(y_{1t+j}, y_{2t+j}, a_{1t+j}(y^{t+j-1}), a_{2t+j}(y^{t+j-1})) dy_{2t+j} dy_{1t+j}\} \\
&= \int \int [u(y_{1t+1} + \tau_{t+1}(y^{t+1})) - a_{1t+1}(y^t)] f(y_{1t+1}, y_{2t+1}, a_{1t+1}(y^t), a_{2t+1}(y^t)) dy_{2t+1} dy_{1t+1} \\
&\quad + \sum_{s=2}^{\infty} \beta^{s-1} \int \dots \int [u(y_{1t+s} + \tau_{t+s}(y^{t+s})) - a_{1t+s}(y^{t+s-1})] d^{2s} F(y^{t+s}; y^t, a_1, a_2).
\end{aligned} \tag{3.39}$$

Also pre-multiplying (3.38) by β gives us

$$\begin{aligned}
&\beta U(y^{t+1}, \tau, a_1, a_2) \\
&= \sum_{s=1}^{\infty} \beta^s \int \dots \int [u(y_{1t+1+s} + \tau_{t+1+s}(y^{t+1+s})) - a_{1t+1+s}(y^{t+s})] d^{2s} F(y^{t+1+s}; y^{t+1}, a_1, a_2)
\end{aligned}$$

$$= \sum_{s=2}^{\infty} \beta^{s-1} \int \dots \int \left[u(y_{1t+s} + \tau_{t+s}(y^{t+s})) - a_{1t+s}(y^{t+s-1}) \right] d^{2(s-1)} F(y^{t+s}; y^{t+1}, a_1, a_2). \quad (3.40)$$

Taking expectations of (3.40) with respect to $t+1$ incomes implies

$$\begin{aligned} & \int \int \beta U(y^{t+1}, \tau, a_1, a_2) f(y_{1t+1}, y_{2t+1}, a_{1t+1}(y^t), a_{2t+1}(y^t)) dy_{2t+1} dy_{1t+1} \\ &= \int \int \sum_{s=2}^{\infty} \beta^{s-1} \int \dots \int \left[u(y_{1t+s} + \tau_{t+s}(y^{t+s})) - a_{1t+s}(y^{t+s-1}) \right] d^{2(s-1)} F(y^{t+s}; y^{t+1}, a_1, a_2) \\ & \quad \times f(y_{1t+1}, y_{2t+1}, a_{1t+1}(y^t), a_{2t+1}(y^t)) dy_{2t+1} dy_{1t+1} \\ &= \sum_{s=2}^{\infty} \beta^{s-1} \int \int \int \dots \int \left[u(y_{1t+s} + \tau_{t+s}(y^{t+s})) - a_{1t+s}(y^{t+s-1}) \right] d^{2s} F(y^{t+s}; y^t, a_1, a_2), \end{aligned} \quad (3.41)$$

since

$$\begin{aligned} & d^{2s} F(y^{t+s}; y^t, a_1, a_2) \\ &= \prod_{j=1}^s \{ f(y_{1t+j}, y_{2t+j}, a_{1t+j}(y^{t+j-1}), a_{2t+j}(y^{t+j-1})) dy_{2t+j} dy_{1t+j} \} \\ &= \prod_{j=2}^s \{ f(y_{1t+j}, y_{2t+j}, a_{1t+j}(y^{t+j-1}), a_{2t+j}(y^{t+j-1})) dy_{2t+j} dy_{1t+j} \} \\ & \quad \times f(y_{1t+1}, y_{2t+1}, a_{1t+1}(y^t), a_{2t+1}(y^t)) dy_{2t+1} dy_{1t+1} \\ &= d^{2(s-1)} F(y^{t+s}; y^{t+1}, a_1, a_2) f(y_{1t+1}, y_{2t+1}, a_{1t+1}(y^t), a_{2t+1}(y^t)) dy_{2t+1} dy_{1t+1}. \end{aligned}$$

So it follows, by substituting (3.41) into (3.39), that

$$\begin{aligned} & U(y^t, \tau, a_1, a_2) \\ &= \int \int \left[u(y_{1t+1} + \tau_{t+1}(y^{t+1})) - a_{1t+1}(y^t) \right] f(y_{1t+1}, y_{2t+1}, a_{1t+1}(y^t), a_{2t+1}(y^t)) dy_{2t+1} dy_{1t+1} \\ & \quad + \beta \int \int U(y^{t+1}, \tau, a_1, a_2) f(y_{1t+1}, y_{2t+1}, a_{1t+1}(y^t), a_{2t+1}(y^t)) dy_{2t+1} dy_{1t+1}. \end{aligned} \quad (3.42)$$

From inspection we see that expression (3.42), evaluated at $t = 0$, yields (3.6). Following the same procedure for the agent's discounted utility yields (3.7).

Proof of Proposition 1. First, derive the partial derivative of the first-best transfer, $\tau^*(w, y_1, y_2)$, with respect to y_1 and y_2 . Differentiate (3.11) with respect to y_1 and y_2 and rearrange to obtain, respectively,

$$\begin{aligned}\frac{\partial \tau^*}{\partial y_1} &= \frac{-\frac{u_{c_1 c_1}(y_1 + \tau^*(w, y_1, y_2))}{u_{c_1}(y_1 + \tau^*(w, y_1, y_2))}}{\frac{u_{c_1 c_1}(y_1 + \tau^*(w, y_1, y_2))}{u_{c_1}(y_1 + \tau^*(w, y_1, y_2))} + \frac{v_{c_2 c_2}(y_2 - \tau^*(w, y_1, y_2))}{v_{c_2}(y_2 - \tau^*(w, y_1, y_2))}} \\ &= -\left(\frac{\rho_1^*}{\rho_1^* + \rho_2^*}\right) \\ &\in [-1, 0]\end{aligned}\tag{3.43}$$

and

$$\begin{aligned}\frac{\partial \tau^*}{\partial y_2} &= \frac{\frac{v_{c_2 c_2}(y_2 - \tau^*(w, y_1, y_2))}{v_{c_2}(y_2 - \tau^*(w, y_1, y_2))}}{\frac{u_{c_1 c_1}(y_1 + \tau^*(w, y_1, y_2))}{u_{c_1}(y_1 + \tau^*(w, y_1, y_2))} + \frac{v_{c_2 c_2}(y_2 - \tau^*(w, y_1, y_2))}{v_{c_2}(y_2 - \tau^*(w, y_1, y_2))}} \\ &= \left(\frac{\rho_2^*}{\rho_1^* + \rho_2^*}\right) \\ &\in [0, 1].\end{aligned}\tag{3.44}$$

It is now straightforward to establish that

$$\frac{\partial c_1^*}{\partial y_1} = \frac{\partial c_1^*}{\partial y_2} = \frac{\rho_2^*}{\rho_1^* + \rho_2^*} \in [0, 1]$$

and

$$\frac{\partial c_2^*}{\partial y_1} = \frac{\partial c_2^*}{\partial y_2} = \frac{\rho_1^*}{\rho_1^* + \rho_2^*} \in [0, 1].$$

■

Proof of Proposition 2. Let $w \in \Lambda$ given. From the Envelope theorem we have

$$U'(w) = \lambda_1^*(w)\tag{3.45}$$

and from expression (3.12) we have

$$U' [V^* (w, y_1, y_2)] = \lambda_1^* (w). \quad (3.46)$$

Combining (3.45) and (3.46) yields

$$U' [V^* (w, y_1, y_2)] = U' (w),$$

or equivalently,

$$w = V^* (w, y_1, y_2),$$

if $U (w)$ is non-linear. Since $w \in \Lambda$ was arbitrarily chosen, the result must hold for all $w \in \Lambda$. ■

Proof of Proposition 3. Let $w \in \Lambda$ given. Combining (3.11) and (3.12) and taking account of the fact that $w = V^* (w, y_1, y_2)$ implies

$$U' (w) = -\frac{u_{c1} (y_1 + \tau^* (w, y_1, y_2))}{v_{c2} (y_2 - \tau^* (w, y_1, y_2))}, \text{ for all } (y_1, y_2) \in [0, \bar{y}]^2. \quad (3.47)$$

Also combining (3.11) and (3.12) and evaluating at $t - 1$ implies

$$U' (w) = -\frac{u_{c1} (y_{1t-1} + \tau_{t-1}^*)}{v_{c2} (y_{2t-1} - \tau_{t-1}^*)}. \quad (3.48)$$

Finally, combining (3.47) and (3.48) yields our result. ■

Proof of proposition 4. Suppose $\frac{f_{ia_i}}{f_i} = k$, constant, for $i = 1, 2$. Since $\int_0^{\bar{y}} f_i dy_i = 1$, for all a_i , we must have $0 = \int_0^{\bar{y}} f_{ia_i} dy_i = \int_0^{\bar{y}} k f_i dy_i = k$ and hence $f_{ia_i} \equiv 0$. But this contradicts assumption 4 that $F_{ia_i} < 0$ for some values of y . So we cannot have $\frac{f_{ia_i}}{f_i}$ constant, $i = 1, 2$. But this just means that as long as we do not have both $\hat{\lambda}_2 (w)$ and $\hat{\lambda}_3 (w)$ being equal to zero, then the right hand side of (3.19) can never be constant and our claim follows. ■

Proof of Lemma 1. From the properties of probability density functions we have

$$\int_0^{\bar{y}} f_i(y_i, a_i(w)) dy_i = 1$$

and

$$\int_0^{\bar{y}} f_{ia_i}(y_i, a_i(w)) dy_i = 0,$$

for all $a_i(w)$ ($i = 1, 2$). But this tells us that there must exist $y_i \in [0, \bar{y}]$ for which $f_{ia_1}(y_i, a_i(w)) < 0$ and $y_i \in [0, \bar{y}]$ for which $f_{ia_1}(y_i, a_i(w)) > 0$. It therefore follows, from continuity, that there must exist $y_i \in [0, \bar{y}]$ for which $f_{ia_1}(y_i, a_i(w)) = 0$. We label this income level $\tilde{y}_i(w)$. To see that $\tilde{y}_i(w)$ is unique under *mlrc*, first note that at $\tilde{y}_i(w)$ we must have $\frac{f_{ia_i}(\tilde{y}_i(w), a_i(w))}{f_i(\tilde{y}_i(w), a_i(w))} = 0$. Then since $f_i(y_i, a_i(w)) > 0$ for all $y_i \in [0, \bar{y}]$, it follows from *mlrc* that for all $y_i < \tilde{y}_i(w)$ we have $f_{ia_i}(y_i(w), a_i(w)) < 0$, and for all $y_i > \tilde{y}_i(w)$ we have $f_{ia_i}(y_i(w), a_i(w)) > 0$. ■

Proof of Lemma 2. Let $w \in \Lambda$ given and let $\tilde{y}_1(w) \in [0, \bar{y}]$ represent the income level for which

$$f_{1a_1}(\tilde{y}_1(w), \hat{a}_1(w)) = 0.$$

Evaluating (3.19) at $\tilde{y}_1(w)$ yields

$$\frac{u_{c_1}(\tilde{y}_1(w) + \hat{\tau}(w, \tilde{y}_1(w), y_2))}{v_{c_2}(y_2 - \hat{\tau}(w, \tilde{y}_1(w), y_2))} = - \left(\hat{\lambda}_1(w) + \hat{\lambda}_2(w) \frac{f_{2a_2}(y_2, \hat{a}_2(w))}{f_2(y_2, \hat{a}_2(w))} \right).$$

Since the marginal utilities are positive, we must have $\left(\hat{\lambda}_1(w) + \hat{\lambda}_2(w) \frac{f_{2a_2}(y_2, \hat{a}_2(w))}{f_2(y_2, \hat{a}_2(w))} \right) < 0$ at $\tilde{y}_1(w)$. However since $\left(\hat{\lambda}_1(w) + \hat{\lambda}_2(w) \frac{f_{2a_2}(y_2, \hat{a}_2(w))}{f_2(y_2, \hat{a}_2(w))} \right)$ is independent of y_1 , then

$$\left(\hat{\lambda}_1(w) + \hat{\lambda}_2(w) \frac{f_{2a_2}(y_2, \hat{a}_2(w))}{f_2(y_2, \hat{a}_2(w))} \right) < 0$$

for all $y_1 \in [0, \bar{y}]$. It follows that we must have

$$\left(1 - \hat{\lambda}_3(w) \frac{f_{1a_1}(y_1(w), \hat{a}_1(w))}{f_1(y_1(w), \hat{a}_1(w))} \right) > 0$$

for all $y_1 \in [0, \bar{y}]$. Repeating the argument for $\tilde{y}_2(w) \in [0, \bar{y}]$, the income level for which

$$f_{2a_2}(\tilde{y}_2(w), \hat{a}_2(w)) = 0,$$

and taking account of $U'(w) = \hat{\lambda}_1(w) < 0$ from the Envelope theorem, implies

$$\left(1 - \hat{\lambda}_3(w) \frac{f_{1a_1}(y_1, \hat{a}_1(w))}{f_1(y_1, \hat{a}_1(w))}\right) > 0,$$

and hence

$$\left(\hat{\lambda}_1(w) + \hat{\lambda}_2(w) \frac{f_{2a_2}(y_2, \hat{a}_2(w))}{f_2(y_2, \hat{a}_2(w))}\right) < 0,$$

for all $y_2 \in [0, \bar{y}]$. Since $w \in \Lambda$ was arbitrarily chosen, the result must hold for all $w \in \Lambda$. ■

Proof of Proposition 5. Holding y_2 constant, differentiate the right hand side of (3.19) with respect to y_1 to establish that this derivative is negative when $\hat{\lambda}_3(w) < 0$. Therefore increases in y_1 reduce $\frac{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}$. Similarly, holding y_1 constant, differentiate the right hand side of (3.19) with respect to y_2 to establish that this derivative is positive when $\hat{\lambda}_2(w) < 0$. Therefore increases in y_2 increase $\frac{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}$. Now note that

$$\frac{\partial \hat{c}_1}{\partial y_1} = \frac{\partial (y_1 + \hat{\tau}(w, y_1, y_2))}{\partial y_1}$$

and

$$\frac{\partial \hat{c}_2}{\partial y_2} = \frac{\partial (y_2 - \hat{\tau}(w, y_1, y_2))}{\partial y_2}.$$

To determine the partial derivative of $\hat{\tau}(w, y_1, y_2)$ with respect to y_1 and y_2 , differentiate (3.19) with respect to y_1 and y_2 and rearrange to obtain, respectively,

$$\frac{\partial \hat{\tau}(w, y_1, y_2)}{\partial y_1} = -\frac{\hat{\rho}_1}{(\hat{\rho}_1 + \hat{\rho}_2)} + \frac{\left(\hat{\lambda}_1 + \hat{\lambda}_2 \frac{f_{2a_2}(y_2, \hat{a}_2(w))}{f_2(y_2, \hat{a}_2(w))}\right) \hat{\lambda}_3 \frac{\partial}{\partial y_1} \left(\frac{f_{1a_1}(y_1, \hat{a}_1(w))}{f_1(y_1, \hat{a}_1(w))}\right) \frac{\hat{v}_{c_2}}{\hat{u}_{c_1}}}{\left(1 - \hat{\lambda}_3 \frac{f_{1a_1}(y_1, \hat{a}_1(w))}{f_1(y_1, \hat{a}_1(w))}\right)^2 (\hat{\rho}_1 + \hat{\rho}_2)} \quad (3.49)$$

and

$$\frac{\partial \hat{\tau}(w, y_1, y_2)}{\partial y_2} = \frac{\hat{\rho}_2}{(\hat{\rho}_1 + \hat{\rho}_2)} + \frac{\hat{\lambda}_2 \frac{\partial}{\partial y_2} \left(\frac{f_{2a_2}(y_2, \hat{a}_2(w))}{f_2(y_2, \hat{a}_2(w))} \right) \frac{\hat{v}_{c_2}}{\hat{u}_{c_1}}}{\left(1 - \hat{\lambda}_3 \frac{f_{1a_1}(y_1, \hat{a}_1(w))}{f_1(y_1, \hat{a}_1(w))} \right) (\hat{\rho}_1 + \hat{\rho}_2)}, \quad (3.50)$$

where $\hat{\rho}_1 = -\frac{u_{c_1 c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}$ and $\hat{\rho}_2 = -\frac{v_{c_2 c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}$ are the coefficients of absolute risk aversion, evaluated at $\hat{\tau}(w, y_1, y_2)$, for the principal and the agent respectively, and $\frac{\hat{v}_{c_2}}{\hat{u}_{c_1}}$ is the marginal utility ratio evaluated at $\hat{\tau}(w, y_1, y_2)$. Taking account of Lemma 2, *mlrc*, $\hat{\lambda}_2(w) < 0$, $\hat{\lambda}_3(w) < 0$ and the fact that $\frac{\hat{\rho}_1}{(\hat{\rho}_1 + \hat{\rho}_2)} \in [0, 1]$ and $\frac{\hat{\rho}_2}{(\hat{\rho}_1 + \hat{\rho}_2)} \in [0, 1]$, we can establish that $\frac{\partial \hat{c}_1}{\partial y_1} > 0$ and $\frac{\partial \hat{c}_2}{\partial y_2} > 0$. Finally it is straightforward to see that the signs of $\frac{\partial \hat{\tau}(w, y_1, y_2)}{\partial y_1}$ and $\frac{\partial \hat{\tau}(w, y_1, y_2)}{\partial y_2}$ are ambiguous. ■

Proof of Proposition 6. Let $w \in \Lambda$ given. From the Envelope theorem we obtain

$$U'(w) = \hat{\lambda}_1(w). \quad (3.51)$$

Also evaluating expression (3.20) at $(\tilde{y}_1(w), \tilde{y}_2(w))$ implies

$$U' \left[\hat{V}(w, \tilde{y}_1(w), \tilde{y}_2(w)) \right] = \hat{\lambda}_1(w). \quad (3.52)$$

Combining (3.51) and (3.52) implies

$$U'(w) = U' \left[\hat{V}(w, \tilde{y}_1(w), \tilde{y}_2(w)) \right],$$

or equivalently,

$$w = \hat{V}(w, \tilde{y}_1(w), \tilde{y}_2(w))$$

since $U(w)$ is non-linear. Evaluate (3.19) at $(\tilde{y}_1(w), \tilde{y}_2(w))$ to get (3.28). Since $w \in \Lambda$ was arbitrarily chosen, the result holds for all $w \in \Lambda$. ■

Proof of Proposition 7. Differentiate (3.20) with respect to y_1 and y_2 and rearrange

to get

$$U'' \left[\hat{V}(w, y_1, y_2) \right] \hat{V}_{y_1}(w, y_1, y_2) = \frac{\left(\hat{\lambda}_1(w) + \hat{\lambda}_2(w) \frac{f_{2a_2}(y_2, \hat{a}_2(w))}{f_2(y_2, \hat{a}_2(w))} \right) \hat{\lambda}_3(w) \frac{\partial}{\partial y_1} \left(\frac{f_{1a_1}(y_1, \hat{a}_1(w))}{f_1(y_1, \hat{a}_1(w))} \right)}{\left(1 - \hat{\lambda}_3(w) \frac{f_{1a_1}(y_1, \hat{a}_1(w))}{f_1(y_1, \hat{a}_1(w))} \right)^2} \quad (3.53)$$

and

$$U'' \left[\hat{V}(w, y_1, y_2) \right] \hat{V}_{y_2}(w, y_1, y_2) = \frac{\hat{\lambda}_2(w) \frac{\partial}{\partial y_2} \left(\frac{f_{2a_2}(y_2, \hat{a}_2(w))}{f_2(y_2, \hat{a}_2(w))} \right)}{\left(1 - \hat{\lambda}_3(w) \frac{f_{1a_1}(y_1, \hat{a}_1(w))}{f_1(y_1, \hat{a}_1(w))} \right)}. \quad (3.54)$$

The result follows by taking account of *mlrc*, $U''(w) < 0$, $\hat{\lambda}_2(w) < 0$, $\hat{\lambda}_3(w) < 0$ and Lemma 2. ■

Proof of proposition 9. Evaluate the first order condition (3.19) at the $t + 1$ utility payoff $w' = \hat{V}(w, \bar{y}_1, \bar{y}_2)$ and re-arrange. This implies

$$\begin{aligned} & -\hat{\lambda}_1(w') \\ = & \frac{u_{c_1}(y'_1 + \hat{\tau}(w', y'_1, y'_2))}{v_{c_2}(y'_2 - \hat{\tau}(w', y'_1, y'_2))} \left(1 - \hat{\lambda}_3(w') \frac{f_{1a_1}(y'_1, \hat{a}_1(w'))}{f_1(y'_1, \hat{a}_1(w'))} \right) + \hat{\lambda}_2(w') \frac{f_{2a_2}(y'_2, \hat{a}_2(w'))}{f_2(y'_2, \hat{a}_2(w'))}. \end{aligned} \quad (3.55)$$

Now notice that

$$\hat{\lambda}_1(w') = U'(w') = U'(\hat{V}(w, \bar{y}_1, \bar{y}_2)) = -\frac{u_{c_1}(\bar{y}_1 + \hat{\tau}(w, \bar{y}_1, \bar{y}_2))}{v_{c_2}(\bar{y}_2 - \hat{\tau}(w, \bar{y}_1, \bar{y}_2))}, \quad (3.56)$$

where the first equality stems from the Envelope Theorem, the second from $w' = \hat{V}(w, \bar{y}_1, \bar{y}_2)$, and the third from conditions (3.19) and (3.20). Combining (3.55) and (3.56) yields

$$\begin{aligned} & \frac{u_{c_1}(\bar{y}_1 + \hat{\tau}(w, \bar{y}_1, \bar{y}_2))}{v_{c_2}(\bar{y}_2 - \hat{\tau}(w, \bar{y}_1, \bar{y}_2))} \\ = & \frac{u_{c_1}(y'_1 + \hat{\tau}(w', y'_1, y'_2))}{v_{c_2}(y'_2 - \hat{\tau}(w', y'_1, y'_2))} \left(1 - \hat{\lambda}_3(w') \frac{f_{1a_1}(y'_1, \hat{a}_1(w'))}{f_1(y'_1, \hat{a}_1(w'))} \right) + \hat{\lambda}_2(w') \frac{f_{2a_2}(y'_2, \hat{a}_2(w'))}{f_2(y'_2, \hat{a}_2(w'))} \end{aligned} \quad (3.57)$$

which simply gives us a relationship between the known marginal utility ratio at time t and the time $t + 1$ marginal utility ratio. This relationship must always be satisfied under the second-best contract. Following a similar procedure for condition (3.20) gives us

$$\begin{aligned} & U' \left[\hat{V}(w, \bar{y}_1, \bar{y}_2) \right] \\ = & U' \left[\hat{V}(w', y'_1, y'_2) \right] \left(1 - \hat{\lambda}_3(w') \frac{f_{1a_1}(y'_1, \hat{a}_1(w'))}{f_1(y'_1, \hat{a}_1(w'))} \right) - \hat{\lambda}_2(w') \frac{f_{2a_2}(y'_2, \hat{a}_2(w'))}{f_2(y'_2, \hat{a}_2(w'))} \end{aligned} \quad (3.58)$$

or a relationship between the time $t + 1$ and the time $t + 2$ utility payoffs. Finally taking account, in conditions (3.57) and (3.58), of *mlrc*, $\hat{\lambda}_2(w') < 0$, $\hat{\lambda}_3(w') < 0$, and the fact that $\left(1 - \hat{\lambda}_3(w') \frac{f_{1a_1}(y'_1, \hat{a}_1(w'))}{f_1(y'_1, \hat{a}_1(w'))} \right)$ is always strictly positive and that $y_i \leq \tilde{y}_i \Leftrightarrow \frac{f_{ia_i}}{f_i} \leq 0$, yields our results. ■

Proof of Proposition 10. Let $w \in \Lambda$ given. Suppose we reverse the roles of the principal and the agent in problem (P) to get a new but equivalent problem given by

$$U(w) = \max_{\{\tau(w, y_1, y_2), V(w, y_1, y_2), a_1(w), a_2(w)\}} Ev \quad (R)$$

subject to $Eu = w$, $\frac{\partial Ev}{\partial a_2} = 0$, and $\frac{\partial Eu}{\partial a_1} = 0$, where this time

$$Eu \equiv \int \int_{[0, \bar{y}]^2} \left[\begin{array}{c} u(y_1 + \tau(w, y_1, y_2)) \\ + \beta V(w, y_1, y_2) \end{array} \right] f(y_1, y_2, a_1(w), a_2(w)) dy_1 dy_2 - a_1(w)$$

and

$$Ev \equiv \int \int_{[0, \bar{y}]^2} \left[\begin{array}{c} v(y_2 - \tau(w, y_1, y_2)) \\ + \beta U(V(w, y_1, y_2)) \end{array} \right] f(y_1, y_2, a_1(w), a_2(w)) dy_1 dy_2 - a_2(w).$$

In (R) the principal and agent now ‘swap’ positions so that the agent (agent v) assumes the role of the principal while the principal (agent u) assumes the role of the agent.

Therefore the principal receives utility payoff w in the current period and $V(w, y_1, y_2)$ in the next period and we maximize the agent's expected discounted utility subject to the principal receiving w today. The agent therefore receives the payoff $U(w)$ today and the payoff $U(V(w, y_1, y_2))$ tomorrow. Because of the symmetry of the current setup the problem (R) must be equivalent to the original problem (P) and hence the optimal contract must possess exactly the same characteristics as our original contract. In the solution to (R) we write $\hat{\lambda}_1(w)$ as the multiplier on the principal's utility promise constraint, $\hat{\lambda}_2(w)$ as the multiplier on the agent's (who now takes the role of principal) incentive compatibility constraint, and $\hat{\lambda}_3(w)$ as the multiplier on the principal's (who now takes the role of agent) incentive compatibility constraint.

From the first order conditions for (R) we have

$$\frac{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))} = - \frac{\left(\hat{\lambda}_1(w) + \hat{\lambda}_3(w) \frac{f_{1a_1}(y_1, \hat{a}_1(w))}{f_1(y_1, \hat{a}_1(w))} \right)}{\left(1 - \hat{\lambda}_2(w) \frac{f_{2a_2}(y_2, \hat{a}_2(w))}{f_2(y_2, \hat{a}_2(w))} \right)} \quad (3.59)$$

and

$$U'(\hat{V}(w, y_1, y_2)) = \frac{\left(\hat{\lambda}_1(w) + \hat{\lambda}_3(w) \frac{f_{1a_1}(y_1, \hat{a}_1(w))}{f_1(y_1, \hat{a}_1(w))} \right)}{\left(1 - \hat{\lambda}_2(w) \frac{f_{2a_2}(y_2, \hat{a}_2(w))}{f_2(y_2, \hat{a}_2(w))} \right)}. \quad (3.60)$$

Note that conditions (3.59) and (3.60) provide exactly the same information as our original conditions (3.19) and (3.20), the only difference being that the terms $\hat{\lambda}_2(w) \frac{f_{2a_2}(y_2, \hat{a}_2(w))}{f_2(y_2, \hat{a}_2(w))}$ and $\hat{\lambda}_3(w) \frac{f_{1a_1}(y_1, \hat{a}_1(w))}{f_1(y_1, \hat{a}_1(w))}$ have 'swapped' positions in the first order conditions reflecting the fact that the roles of the principal and the agent have now been reversed. Apart from this nothing else has changed. We now take account of the first order conditions from the original problem (P) and the reversed problem (R) in obtaining our result. Applying

the Envelope theorem to (P) implies

$$U'(w) = \hat{\lambda}_1(w) \quad (3.61)$$

and re-arranging (3.19) implies

$$-\hat{\lambda}_1(w) = \frac{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))} \left(1 - \hat{\lambda}_3(w) \frac{f_{1a_1}(y_1, \hat{a}_1(w))}{f_1(y_1, \hat{a}_1(w))} \right) + \hat{\lambda}_2(w) \frac{f_{2a_2}(y_2, \hat{a}_2(w))}{f_2(y_2, \hat{a}_2(w))}. \quad (3.62)$$

Combining (3.61) and (3.62) and taking expectations on both sides implies¹²

$$\begin{aligned} U'(w) &= -\frac{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))} \left(1 - \hat{\lambda}_3(w) \frac{f_{1a_1}(y_1, \hat{a}_1(w))}{f_1(y_1, \hat{a}_1(w))} \right) - \hat{\lambda}_2(w) \frac{f_{2a_2}(y_2, \hat{a}_2(w))}{f_2(y_2, \hat{a}_2(w))} \\ &= -E \left\{ \frac{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))} \left(1 - \hat{\lambda}_3(w) \frac{f_{1a_1}(y_1, \hat{a}_1(w))}{f_1(y_1, \hat{a}_1(w))} \right) \right\}. \end{aligned} \quad (3.63)$$

Conditions (3.19) and (3.20), evaluated at $t - 1$, yield

$$U'(w) = -\frac{u_{c_1}(y_{1t-1} + \hat{\tau}_{t-1})}{v_{c_2}(y_{2t-1} - \hat{\tau}_{t-1})}, \quad (3.64)$$

and combining (3.63) and (3.64) gives us

$$\frac{u_{c_1}(y_{1t-1} + \hat{\tau}_{t-1})}{v_{c_2}(y_{2t-1} - \hat{\tau}_{t-1})} = E \left\{ \frac{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))} \left(1 - \hat{\lambda}_3(w) \frac{f_{1a_1}(y_1, \hat{a}_1(w))}{f_1(y_1, \hat{a}_1(w))} \right) \right\}. \quad (3.65)$$

¹²Under the assumption that individual incomes are independent random variables we have $f = f_1 f_2$ and hence $f_{a_2} = f_1 f_{2a_2}$. This implies $\frac{f_{a_2}}{f} = \frac{f_{2a_2}}{f_2}$ and so taking expectations implies

$$\begin{aligned} \int \int_{[0, \bar{y}]^2} \frac{f_{2a_2}}{f_2} f dy_1 dy_2 &= \int \int_{[0, \bar{y}]^2} \frac{f_{a_2}}{f} f dy_1 dy_2 \\ &= \int \int_{[0, \bar{y}]^2} f_{a_2} dy_1 dy_2 \\ &= 0. \end{aligned}$$

Note that this can also be expressed as

$$\begin{aligned} \frac{u_{c_1}(y_{1t-1} + \hat{\tau}_{t-1})}{v_{c_2}(y_{2t-1} - \hat{\tau}_{t-1})} &= E \left\{ \frac{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))} \right\} \\ &\quad - \hat{\lambda}_3(w) Cov \left\{ \frac{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}, \frac{f_{1a_1}(y_1, \hat{a}_1(w))}{f_1(y_1, \hat{a}_1(w))} \right\}. \end{aligned} \quad (3.66)$$

Now apply the Envelope theorem to (R) to obtain (3.61) and re-arrange (3.59) to get

$$-\hat{\lambda}_1(w) = \frac{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))} \left(1 - \hat{\lambda}_2(w) \frac{f_{2a_2}(y_2, \hat{a}_2(w))}{f_2(y_2, \hat{a}_2(w))} \right) + \hat{\lambda}_3(w) \frac{f_{1a_1}(y_1, \hat{a}_1(w))}{f_1(y_1, \hat{a}_1(w))}. \quad (3.67)$$

Combining (3.61) and (3.67) and taking expectations on both sides implies

$$U'(w) = -E \left\{ \frac{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))} \left(1 - \hat{\lambda}_2(w) \frac{f_{2a_2}(y_2, \hat{a}_2(w))}{f_2(y_2, \hat{a}_2(w))} \right) \right\}. \quad (3.68)$$

Conditions (3.59) and (3.60), evaluated at $t - 1$, yield

$$U'(w) = -\frac{v_{c_2}(y_{2t-1} - \hat{\tau}_{t-1})}{u_{c_1}(y_{1t-1} + \hat{\tau}_{t-1})}, \quad (3.69)$$

and combining (3.68) and (3.69) gives us

$$\frac{v_{c_2}(y_{2t-1} - \hat{\tau}_{t-1})}{u_{c_1}(y_{1t-1} + \hat{\tau}_{t-1})} = E \left\{ \frac{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))} \left(1 - \hat{\lambda}_2(w) \frac{f_{2a_2}(y_2, \hat{a}_2(w))}{f_2(y_2, \hat{a}_2(w))} \right) \right\}. \quad (3.70)$$

Again this can be expressed as

$$\begin{aligned} \frac{v_{c_2}(y_{2t-1} - \hat{\tau}_{t-1})}{u_{c_1}(y_{1t-1} + \hat{\tau}_{t-1})} &= E \left\{ \frac{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))} \right\} \\ &\quad - \hat{\lambda}_2(w) Cov \left\{ \frac{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}, \frac{f_{2a_2}(y_2, \hat{a}_2(w))}{f_2(y_2, \hat{a}_2(w))} \right\}. \end{aligned} \quad (3.71)$$

Since the problems (P) and (R) are equivalent this means that conditions (3.66) and (3.71) must both characterize the optimal contract under repeated double moral hazard. Now recall from proposition 5 we had $\frac{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}$ decreasing in y_1 whenever $\hat{\lambda}_3(w) < 0$. It is also straightforward to see that $\frac{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}$ must be increasing in y_1 whenever $\hat{\lambda}_3(w) > 0$. Also under the *mlrc* we have $\frac{f_{1a_1}(y_1, \hat{a}_1(w))}{f_1(y_1, \hat{a}_1(w))}$ increasing in y_1 . Therefore whenever $\hat{\lambda}_3(w) < 0$ we must have $Cov \left\{ \frac{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}, \frac{f_{1a_1}(y_1, \hat{a}_1(w))}{f_1(y_1, \hat{a}_1(w))} \right\} < 0$. Likewise if $\hat{\lambda}_3(w) > 0$ then we must have $Cov \left\{ \frac{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}, \frac{f_{1a_1}(y_1, \hat{a}_1(w))}{f_1(y_1, \hat{a}_1(w))} \right\} > 0$. Therefore condition (3.66) states that whenever $\hat{\lambda}_3(w) \neq 0$ we must have

$$\frac{u_{c_1}(y_{1t-1} + \hat{\tau}_{t-1})}{v_{c_2}(y_{2t-1} - \hat{\tau}_{t-1})} < E \left\{ \frac{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))} \right\}, \quad (3.72)$$

or equivalently

$$\frac{v_{c_2}(y_{2t-1} - \hat{\tau}_{t-1})}{u_{c_1}(y_{1t-1} + \hat{\tau}_{t-1})} > \frac{1}{E \left\{ \frac{1}{\frac{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}} \right\}}, \quad (3.73)$$

which gives us the first inequality in (3.33). Now differentiate the right hand side of (3.59) with respect to y_2 to see that $\frac{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}$ is decreasing in y_2 whenever $\hat{\lambda}_2(w) < 0$ and increasing in y_2 whenever $\hat{\lambda}_2(w) > 0$. (Remember that $\left(\hat{\lambda}_1(w) + \hat{\lambda}_3(w) \frac{f_{1a_1}(y_1, \hat{a}_1(w))}{f_1(y_1, \hat{a}_1(w))} \right)$ must always be negative). Also under the *mlrc* we have $\frac{f_{2a_2}(y_2, \hat{a}_2(w))}{f_2(y_2, \hat{a}_2(w))}$ increasing in y_2 . It follows that whenever $\hat{\lambda}_2(w) < 0$ we have $Cov \left\{ \frac{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}, \frac{f_{2a_2}(y_2, \hat{a}_2(w))}{f_2(y_2, \hat{a}_2(w))} \right\} < 0$ and whenever $\hat{\lambda}_2(w) > 0$ we have

$$Cov \left\{ \frac{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))}, \frac{f_{2a_2}(y_2, \hat{a}_2(w))}{f_2(y_2, \hat{a}_2(w))} \right\} > 0.$$

Therefore condition (3.71) states that as long as $\hat{\lambda}_2(w) \neq 0$ we must have

$$\frac{v_{c_2}(y_{2t-1} - \hat{\tau}_{t-1})}{u_{c_1}(y_{1t-1} + \hat{\tau}_{t-1})} < E \left\{ \frac{v_{c_2}(y_2 - \hat{\tau}(w, y_1, y_2))}{u_{c_1}(y_1 + \hat{\tau}(w, y_1, y_2))} \right\} \quad (3.74)$$

which gives us the second inequality in (3.33). Finally, from (3.66) and (3.71) we see that if $\hat{\lambda}_3(w) = 0$ and $\hat{\lambda}_2(w) \neq 0$ then (3.32) must hold, while if $\hat{\lambda}_2(w) = 0$ and $\hat{\lambda}_3(w) \neq 0$

then (3.31) must be true. Since $w \in \Lambda$ was arbitrarily chosen these results must hold for all $w \in \Lambda$. ■

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Chapter 4

Family Insurance and the Welfare State: The Role of Public Insurance Schemes in the Presence of Family Risk-Sharing Arrangements

4.1 Background and Literature Review

An established literature in labour economics and public finance explores questions related to the provision of unemployment insurance and the effect this has on the rate of unemployment in the economy (see Baily, 1978; Ehrenberg and Oaxaca, 1976; Feldstein, 1976, 1978; Nickell, 1979; and Shavell and Weiss, 1979). A parallel literature in public finance and macroeconomics investigates the appropriate role for the redistributive tax system as a device for reducing the risks associated with idiosyncratic income fluctuations (see Varian, 1980). However one criticism of this literature is based on the observation that the state is by no means the only source of insurance available to individuals facing volatile income streams and that there are alternative institutions that also play a key role in helping individuals to achieve a smoother consumption profile. We find that many

models of unemployment insurance do not take account of the income earned by other members of a claimant's household, nor of the important insurance role played by institutions such as the family.¹ However as we demonstrate in this chapter the development of appropriate policy initiatives will require taking account of the role played by such institutions.

There is now growing empirical evidence that the family is an important source of informal insurance and that various forms of risk-sharing arrangements exist between family members (see Lucas and Stark, 1985; Rosenzweig, 1988; Rosenzweig and Stark, 1988). A number of papers assume that, in making intra-family income transfers, family members are *altruistic* (e.g. Barro, 1974; Becker, 1974, 1991; and Ravallion and Dearden, 1988). Under this assumption one benevolent individual cares about the well-being of another and makes income transfers even when there are no penalties for failing to do so. However several authors question such altruistic motivations and indeed find evidence that observed patterns of intra-family income transfers (Cox, 1987), or of bequests (Bernheim, Shleifer and Summers, 1985), are more consistent with exchange-related motives, i.e. income transfers are just one part of a broader transaction containing some *quid pro quo*.

In modelling intra-family insurance arrangements based on exchange-related motives, household members are regarded as self-interested and risk averse and so insurance arrangements exist because such arrangements are mutually beneficial. Within this tradition there have been two main strands in the literature. Under the first strand family members are able to pre-commitment to upholding their part in the risk-sharing arrangement (e.g. Arnott and Stiglitz, 1991). Under the second strand such pre-commitments are not possible and so insurance arrangements can be sustained only through reciprocity in a repeated relationship (e.g. Coate and Ravallion, 1993; Kimball, 1988; and Ligon, Thomas and Worrall, 1997). In this case risk averse agents receive random incomes each

¹Indeed Atkinson and Micklewright (1991) also argue that the theoretical literature on unemployment benefit largely ignores many of the key institutional features of actual social security systems.

period and any insurance arrangement will be self-sustaining as long as the long-term value to each person of remaining within the arrangement exceeds any short-term benefit they might derive from leaving the arrangement.

Another more recent approach to income-sharing has been to suggest that such arrangements are sustained through the existence of *social norms* (see, for instance, Ambec, 2000). This has been suggested as one possible explanation as to why otherwise selfish individuals may make income transfers even when they do not expect a *quid pro quo*, or why they sometimes share their income with people who are not part of their immediate family, such as members of their extended family, village or kinship group. For example Lucas and Stark (1985) provide evidence for the existence of migrants' remittances in the absence of reverse income flows, but argue that neither pure altruism nor pure self-interest are likely to adequately describe the more complex and subtle forms of interaction that actually take place between family members. Also Fafchamps (1995) observes that terminally ill and mentally/physically disabled members of a community are not always excluded from mutual assistance networks. In the former case reciprocity is unlikely to explain the existence of migrants' remittances, while in the latter case altruistic feelings may well be very low and there is also little chance of reciprocity. Under the social norm approach proposed by Ambec, an income-sharing norm is designed and implemented by elders, and mutually enforced by members of the community.

Throughout this chapter we take the existence of intra-family insurance arrangements as given and explore the consequences for such arrangements of also providing income insurance through a welfare state. We also examine questions to do with the optimal design of the welfare state in the presence of intra-family insurance arrangements. In particular we address three main questions. First, what is the effect of an expansion of the welfare state on the level of income transferred between family members and on the overall level of insurance available to individuals? Second, what are the mechanisms through which family income transfers are affected? Third, what role exists for a public insurance scheme to enhance welfare when insurance is also provided within family networks?

Our analysis is based on the simple exchange approach to insurance, applied to risk-sharing arrangements between members of a family. Families consist of two risk averse individuals, H and W , who make credible commitments to each other to honour a risk-sharing arrangement. Both H and W receive a stochastic income and interaction takes place only once. Within this framework unemployed family members receive public transfers from the state while employed family members pay taxes to the state. We assume that in taking their private decisions individuals regard the level of the tax and the public transfer as given. Nevertheless it is the collective actions of all persons in the economy, as well as the presence of insurance arrangements within family networks, that affect the socially optimal level of the tax and transfer.

We carry out our analysis under three different scenarios. Under the first and simplest scenario, the income distribution faced by each individual is exogenously given. This allows us to abstract from the effect a person's actions might have on their chances of being employed, and from any moral hazard and informational problems that arise as a result. Under this simple scenario the role of state-sponsored public insurance schemes in raising social welfare, even when insurance arrangements exist within family networks, is thrown into stark relief. Although increases in the size of the public transfer cause an unambiguous reduction in the level of income transferred between family members, a social welfare maximum entails the state providing full insurance to individuals combined with the complete elimination of insurance between family members. The intuition behind this result is based on the fact that family insurance occurs within a small group of just two risk averse persons whereas a risk neutral government can implement income transfers across a much larger group of people. Since there are no incentive problems the government is able to offer each individual a guaranteed level of consumption across each state of the world, thereby eliminating any consumption variation while preserving the level of expected consumption. This must improve the welfare of any risk averse individual.

Under this simple scenario we also shed some light on the *extent* of the crowding out

of family transfers caused by an increase in the size of the welfare state. Suppose p is the probability that a family member is employed while $(1 - p)$ is the probability that s/he is unemployed. Then we show that an increase in the level of the public transfer will crowd out family transfers exactly one-for-one *if and only if* $p = \frac{1}{2}$. Also an increase in the level of the public transfer will crowd out family transfers by more than (less than) one-for-one *if and only if* $p < \frac{1}{2}$ ($p > \frac{1}{2}$). The intuition for this result is straightforward. First note that p can also be interpreted as the proportion of employed individuals in the economy, with $(1 - p)$ representing the proportion of unemployed individuals. Also note that under this simple scenario individuals will always offer full insurance to their family members, regardless of the level of the tax and public transfer. When the proportion of employed individuals is large (say $p > \frac{1}{2}$), a given increase in the size of the public transfer paid to the unemployed will require a relatively small increase in the tax paid by the employed to support the relatively small numbers of unemployed. Therefore for any given increase in the level of the public transfer, the reduction in the net income of employed individuals will be relatively small. But since the family transfer is just half the difference between family members' realized net incomes, then such changes to the tax and public transfer will have a relatively small crowding out effect on the family transfer. On the other hand, when the proportion of employed individuals is small (say $p < \frac{1}{2}$), then any increase in the size of the public transfer paid to the unemployed will require a relatively large increase in the tax paid by employed individuals to support the relatively large numbers of unemployed. This time as public transfers increase, the reduction in the net income of the employed will be relatively large and so the crowding out effect on the family transfer must also be large.

Under the second scenario we assume that individuals' income distributions are endogenously determined by their level of effort. We also assume that the government is unable to observe individuals' effort choices but that family members are in a position to make such observations. Under this scenario we show that unless effort incentives are such that an increase in the transfer received by the unemployed requires, for balancing

the budget, a *reduction* in the level of the tax paid by the employed *by an equal or greater amount* than the increase in the transfer, then increases in the public transfer must lead to a reduction in the level of the family income transfer. Additionally and in keeping with standard results from the insurance and moral hazard literature we also show that when the government cannot observe individual actions, any social welfare maximum must entail the state providing less than full insurance to individuals. This is true even when insurance is provided within family networks. Finally we also show that under this scenario, despite any crowding out of family transfers by increases in the size of the welfare state, there remains a clear role for public insurance schemes to improve welfare beyond the level that would be achieved if only family insurance were available. To do so we demonstrate that when insurance is available within family networks, a social welfare maximum can never entail zero public insurance.

Under our third and final scenario, income distributions remain dependent on individual effort and we preserve the assumption that the government is unable to observe individual actions. However this time we assume that family members are also unable to observe each others actions. We therefore have a problem of moral hazard both between the government and family, as well as within each family. In this case the family income transfer depends not only on the level of taxes and public transfers but also on the family members' effort levels. This contrasts with both our previous scenarios under which family members offered each other full insurance and the level of the family transfer was independent of the level of effort. Under the current scenario we argue that any feasible expansion of the welfare state must have two effects on the intra-family income transfer. First, changes to the public insurance scheme affect family members' net incomes, thereby affecting each person's variation in net incomes across states of the world. Such changes also affect the difference between family members net incomes in any given state of the world. Therefore changes to the public insurance scheme affect the level of the intra-family income transfer by affecting the scope for risk-sharing between family members. Second, changes to the tax schedule also affect the level of effort chosen by family

members and this in turn will exert an additional effect on the level of the intra-family income transfer. Therefore when there is moral hazard within the family, the overall impact of a change in the size of the welfare state on the family income transfer must depend on the combination of both these effects.

Under the third scenario we also show that the optimal welfare state must offer less than full insurance to individuals. As before, given the government's inability to observe individual actions, this result is in keeping with standard results in insurance and moral hazard. However this time the case for having any public insurance scheme at all is not so clear cut. Recall that under the second scenario when effort was observable within the family, we demonstrated that a social welfare maximum could never entail zero public insurance. In that case some degree of government intervention was always optimal. However under the third scenario we identify conditions under which increasing the size of the welfare state, from a position of no government intervention, will be welfare reducing. Under these conditions a social welfare maximum exists locally at the point of no government intervention. This means that for sufficiently small levels of government intervention there is no scope for improving welfare via the redistributive tax system, and so there is a social welfare (local) maximum that entails the provision of no public insurance. An alternative interpretation of this result is that for sufficiently small levels of government intervention it is preferable for the government to offer no public insurance at all. We show that the occurrence of this outcome will depend on the sign and magnitude of the effects of changes in the welfare state on effort and on the family income transfer.

Several papers also examine different aspects of the relationship between insurance provided within the family and the insurance available from non-family sources such as the market or the state. For example Arnott and Stiglitz (1991) examine the interaction between market and nonmarket (i.e. family) insurance. In their analysis individuals acquire market insurance from insurers who operate in a perfectly competitive environment. However because of moral hazard individuals receive less than full insurance from the market insurers. Arnott and Stiglitz show that if individuals form risk-sharing arrange-

ments with family members once they have purchased market insurance, such family insurance can be unambiguously harmful in welfare terms. Acquiring additional family insurance can worsen effort incentives, causing market insurers to restrict even further the level of insurance they provide. Under such circumstances family insurance between risk averse agents crowds out the market insurance provided by a risk neutral firm, and this must be welfare reducing.

Arnott and Stiglitz essentially compare welfare consequences across different insurance arrangements and assumptions about the observability of effort. They establish that

$$EU^1 > EU^{NMO} > EU^M > EU^{NMU},$$

where EU^1 is expected utility when effort is observable to both market and family (non-market) insurers, i.e. expected utility under the first-best scenario, EU^{NMO} is expected utility when family insurance exists and effort is observable within the family but not by the market insurer, EU^M is expected utility when only market insurance is available and effort is unobservable by the market insurer (i.e. expected utility when family insurance is not available), and EU^{NMU} is expected utility when family insurance exists and effort is unobservable both within the family and by the market insurer. Based on their analysis the authors suggest that if effort is unobservable within families, then “.....the equilibrium without nonmarket insurance cannot be improved upon, and if it were possible, it would be desirable to outlaw the provision of nonmarket insurance.” (p.186).

The analysis of this chapter is based on the same framework as that developed by Arnott and Stiglitz in which we replace their competitive insurance markets by a state-provided public insurance scheme. However, although we adopt their basic framework, we nevertheless pursue a very different line of inquiry. Whereas Arnott and Stiglitz ask ‘what happens to social welfare when family members, who have acquired market insurance, also form risk-sharing arrangements?’, throughout this chapter we take the existence of family insurance arrangements to be given and ask ‘what effect does the provision of public insurance (as an additional source of insurance) have on the level of income

transferred between family members, on the overall insurance available to individuals, and on social welfare, *vis-a-vis* the situation with no public insurance?’ We also ask ‘how does the existence of family insurance arrangements affect the optimal design of the welfare state?’ We believe ours is a necessary line of inquiry, particularly in view of the fact that, as a policy initiative, outlawing the provision of nonmarket (especially family) insurance is likely to be (a) prohibitively costly to enforce, (b) ultimately futile, and (c) likely to raise serious ethical questions surrounding the appropriate level of state intervention into the economic arrangements between members of a family.

Di Tella and MacCulloch (1998) examine the effect of public insurance schemes on transfers between family members and on the total insurance available to family members, as well as questions related to the optimal design of the welfare state. However it is interesting to find that the results of their analysis are in stark contrast to ours. Under the assumption of enforceable family contracts and no moral hazard (note that the analysis of our first scenario rests upon identical assumptions), Di Tella and MacCulloch show that public transfers crowd out family transfers exactly one-for-one and that the provision of public insurance, when insurance is also provided within the family, does not affect the total insurance available to family members. Note that this contradicts our own findings that (a) the extent to which family transfers are crowded out by an expansion of the welfare state will depend entirely upon the proportion of employed and unemployed individuals in the economy, and (b) until the full insurance point is reached, more public insurance reduces individuals’ consumption variance, thereby increasing their overall level of insurance as well as their level of welfare. Di Tella and MacCulloch also establish that the size of the welfare state is irrelevant to social welfare. The upshot of their analysis is that it does not matter whether family members obtain their insurance from within the family or from the state. Again this contrasts strongly with our own findings that (a) the size of the welfare state matters in welfare terms, and (b) maximizing social welfare entails the state providing full insurance and family members providing none. We believe that the key explanation for these differences in results is that whereas Di

Tella and MacCulloch assume large families, thereby diminishing the state's advantage in pooling risk, we have assumed that families are small, thus enhancing the capacity of the state to improve welfare by pooling risks *across* families.

Other papers, in examining the relationship between state-sponsored insurance schemes and family insurance arrangements, focus on the situation where family contracts are non-binding. We now summarize some of the main results from these papers. Krueger and Perri (1999) regard the tax system as a form of forced risk-sharing and assume that private contracts can be enforced only by the threat of exclusion from future credit markets. They examine the welfare consequences of different taxation schemes when agents participate in such credit markets, and demonstrate that changes in the tax system can reduce the severity of punishment for agents who default. This in turn reduces the incentives that agents face to enter into private risk-sharing contracts, and so restricts the set of such contracts that it might be possible to sustain². They also show that the welfare consequences of such changes depend on the extent to which private insurance declines in response to the increased risk-sharing forced via the tax system. The authors calibrate an artificial economy to United States income and tax data and show that for plausible values of the structural parameters, a more redistributive tax system leads to less risk-sharing among individuals and to lower *ex-ante* welfare.

The paper by Di Tella and MacCulloch (1998) referred to earlier also explores a similar set of questions by arguing that the state can provide compulsory insurance through the taxation of individuals whereas the family must rely on self-enforcing agreements. Under this assumption, increases in the level of unemployment benefit crowd out family income

²A number of empirical studies (Attanasio and Davis, 1996; Hayashi *et al*, 1996) have challenged theories of complete insurance markets by showing that perfect risk-sharing among individuals does not exist. In response to this there is a large literature that attempts to incorporate market incompleteness into the economic environment. One approach has been to assume market incompleteness from the outset (see Bewley, 1986; Kimball and Mankiw, 1989; Huggett, 1993; and Aiyagari, 1994). Another has been to generate market incompleteness based on informational problems such as moral hazard and adverse selection (see Cole and Kocherlakota, 1998). A third approach, however, has been to suggest that the availability of compulsory insurance, via the tax system, may actually reduce the incentives individuals face for entering into private insurance contracts.

transfers more than one-for-one. The intuition for this is as follows. As the state increases the level of unemployment benefit, families reduce their transfers one-for-one as they try to return to the initial level of insurance. However, the increased generosity of state benefits also makes defecting from family contracts more attractive. Hence family transfers must be reduced even further if family insurance contracts are to remain incentive compatible. State-provided insurance therefore changes the opportunity cost of belonging to a family and the punishment for those who default on family agreements, thereby changing the amount of insurance families can sustain. Under these circumstances the authors show that there still remains a role for state-provided insurance. This time the optimal level of insurance will be related to the natural strength of families in enforcing contracts. When moral hazard is introduced into this scenario and it is assumed that families are more effective at monitoring the actions of its members than the state, then more public insurance crowds out family insurance more than one-for-one. In addition to increasing the attractiveness of defecting from family contracts, the disincentive effects of increased state insurance leads to higher unemployment and increases the tax burden of employed family members to support the greater numbers of unemployed. This increases their willingness to default even further, thereby adding to the reduction in family transfers. Concerning the optimal role for state-insurance under such circumstances, Di Tella and MacCulloch (1998) show that if families are sufficiently powerful at enforcing agreements, then there is no role for the state in providing insurance. Otherwise, the state should be the sole provider of insurance.

Finally, in a couple of papers by Attanasio and Ríos-Rull (1999a, 1999b), the authors also examine the effects of public compulsory insurance arrangements on private insurance with limited commitment. They show that such insurance improves the alternatives to private insurance, thereby reducing the discipline that enforces private insurance arrangements, and can both improve welfare as well as reduce it. The authors use data on the Mexican Progresa program to document the impact that government programs can have in crowding out private transfers.

This chapter is organized as follows. In section 4.2 we present a model of family insurance under a welfare state when income distributions are exogenous, family contracts are enforceable and there are no moral hazard problems, and we analyze the results for this case. In section 4.3 we extend this basic model to take account of individual effort. Here we obtain results under the assumptions that the government cannot observe individual actions but that family members can, and that family contracts are enforceable. In section 4.4 we relax the assumption that family members can observe individual actions and assume instead that there is moral hazard within families. Also we retain the assumption that family contracts are enforceable. Finally in section 4.5 we present some areas for further work.

4.2 A Model of Family Insurance under a Welfare State: Exogenous Income Distributions

We assume that there are n identical families in the economy, each family consisting of two members $i = H, W$. H and W each receive a stochastic *gross* income drawn from the finite set $\{0, Y\}$. The probability of receiving Y is p , while the probability of receiving zero is $(1 - p)$. We introduce the simplest form of a welfare state as follows. Let $\tau > 0$ represent the government transfer received by a family member whenever his/her income is zero, and $0 < r < Y$ represent the tax paid by a family member whenever his/her income is Y . Throughout this chapter we will refer to the values $\{\tau, r\}$ as either a welfare state, a public insurance scheme, a schedule of redistributive taxes and transfers, or simply a tax schedule. Any tax schedule $\{\tau, r\}$ must always satisfy the government's budget constraint

$$pr - (1 - p)\tau = 0, \quad (4.1)$$

which simply states that *expected* deficits or surpluses must equal zero. Note that since p can also be interpreted as the *proportion* of the $2n$ individuals in the economy who are

employed and receive income Y (and $(1 - p)$ as the proportion who are unemployed and receive zero income), then (4.1) can also be interpreted as saying that the government's total tax revenue must always equal the total transfer payments made. Under this interpretation condition (4.1) says that there can be no *actual* budget deficit or surplus. Under the tax schedule $\{\tau, r\}$, H and W each receive stochastic *net* incomes drawn from the finite set $\{\tau, Y - r\}$, with the probability of receiving τ given by p and the probability of receiving $Y - r$ given by $(1 - p)$.

Family members are assumed to be risk averse and to have identical, differentiable, state-independent utility functions. Let $F_{y', y''}$ represent the income transferred from the person who receives *net* income $y' \in \{\tau, Y - r\}$ to the person who receives *net* income $y'' \in \{\tau, Y - r\}$. First note that by symmetry, whenever both family members receive the same net income, there will be no income transfer within the family. Therefore $F_{\tau, \tau} = F_{Y-r, Y-r} = 0$. Second, whenever family members receive different net incomes, then combining symmetry with the fact that income must always be transferred from high to low income individuals³ gives us $-F_{\tau, Y-r} = F_{Y-r, \tau} = F > 0$ when $Y - r > \tau$, and $-F_{\tau, Y-r} = F_{Y-r, \tau} = F < 0$ when $Y - r < \tau$. We can therefore write each family member's expected utility as

$$EU = (1 - p)^2 u(\tau) + (1 - p) p u(\tau + F) + p(1 - p) u(Y - r - F) + p^2 u(Y - r),$$

where $u(\cdot)$ represents the utility function of each family member. We assume $u' > 0$ and $u'' < 0$. We restrict family members' consumption in any state to be strictly positive and so we must have $0 < F < Y - r$ whenever $Y - r > \tau$, and $0 > F > -\tau$ whenever

³Since the probability that an individual receives an income transfer equals the probability that s/he pays an income transfer of the same amount, then under the current setup with constant probabilities, an individual's expected consumption will be independent of the level of the income transfer, F . Therefore if positive transfers occurred from low to high income individuals, then such an intra-family insurance arrangement would *increase* individual consumption variance while having no impact on expected consumption, thereby making risk averse agents worse off. When income distributions are endogenous and probabilities depend on individual effort, this argument holds as long as each person chooses the same effort level and therefore faces the same probability distribution of income.

$Y - r < \tau$. Finally we assume that any intra-family insurance arrangement made *ex ante* will be binding *ex post*.

In choosing the family transfer family members take the tax schedule $\{\tau, r\}$ as given and choose F to

$$\max_F \left\{ (1-p)^2 u(\tau) + (1-p)p u(\tau + F) + p(1-p) u(Y - r - F) + p^2 u(Y - r) \right\}.$$

It is easy to see that the first order condition for this problem implies $\frac{u'(\tau+F)}{u'(Y-r-F)} = 1$ or $F = \frac{Y-r-\tau}{2}$. Therefore for any tax schedule $\{\tau, r\}$, when income distributions are exogenous and family insurance contracts are enforceable then family members provide full insurance to each other. Family members pool their incomes (net of taxes and transfers) so that in any state, each family member consumes exactly half of the total net family income *in that state*.

We are interested in examining the effect of a *feasible* expansion in the welfare state on the level of insurance provided within the family.

Definition 1. A *feasible expansion of the welfare state* is an increase in the public transfer, τ , when such an increase is combined with the change in the tax, r , that would be required in order to maintain a balanced budget.

In this simple framework since the budget constraint stipulates $pr - (1-p)\tau = 0$, where p is constant, then as the welfare state increases in size changes in τ and r must satisfy

$$\frac{dr}{d\tau|_{B1}} = \frac{(1-p)}{p} \quad (4.2)$$

in order for such an expansion to be feasible. This simply says that for every dollar increase in the level of the transfer, τ , paid out to unemployed individuals, the government must also increase the tax, r , that it receives from employed individuals by an amount equal to $\frac{(1-p)}{p}$, in order for the budget to remain in balance.

We can now state the following result.

Proposition 1. When there are no moral hazard problems and family insurance

contracts are enforceable, any feasible expansion of the welfare state must unambiguously crowd out family transfers. Furthermore the extent to which family transfers are crowded out depends on the size of p .

Proof. Differentiate $F = \frac{Y-r-\tau}{2}$ with respect to τ , taking account of the change in r that is required to maintain a balanced budget. This implies

$$\frac{dF}{d\tau} = \frac{\partial F}{\partial \tau} + \frac{\partial F}{\partial r} \frac{dr}{d\tau|_{B1}} = -\frac{1}{2p} < 0. \quad (4.3)$$

■

Expression (4.3) establishes that under our simple framework, any feasible expansion of the welfare state will always lead to a reduction in the amount of income transferred between family members. It also demonstrates that the extent to which the family income transfer decreases in response to the increased generosity of the state will depend upon the size of p . We can now state the following corollary.

Corollary 1. Any feasible expansion of the welfare state will crowd out family transfers exactly one-for-one *if and only if* $p = \frac{1}{2}$. Also any feasible expansion of the welfare state will crowd out family transfers more than (less than) one-for-one *if and only if* $p < \frac{1}{2}$ ($p > \frac{1}{2}$). Specifically we have

$$\frac{dF}{d\tau} = -1 \Leftrightarrow p = \frac{1}{2},$$

$$-1 < \frac{dF}{d\tau} < 0 \Leftrightarrow p > \frac{1}{2}$$

and

$$\frac{dF}{d\tau} < -1 \Leftrightarrow p < \frac{1}{2}.$$

Proof. These results follow directly from (4.3). ■

The intuition for this result is straightforward. When the proportion, p , of employed

individuals in the economy is large (say $p > \frac{1}{2}$), then an increase in the public transfer, τ , will require a relatively small increase in the tax, r , in order for the budget to remain in balance. Therefore as τ increases, the reduction in the net income of employed individuals (i.e. the reduction in $Y - r$) will be relatively small. Since the intra-family income transfer is just half the difference between family members' net incomes, such a change to the tax schedule will have only a small crowding out effect on the level of F . On the other hand, when the proportion, p , of employed individuals is small (say $p < \frac{1}{2}$), then increases in τ will require a relatively large increase in the tax, r , to maintain a balanced budget. This time as τ increases, the reduction in the net income of employed individuals will be relatively large and the crowding out effect on F must also be large.

Note that it is important not to confuse a reduction in the level of F due to a change in the tax schedule as necessarily implying a reduction in the overall amount of insurance available to family members. Suppose we take an increase in the total level of insurance to be equivalent to a reduction in individuals' consumption variance, with expected consumption remaining unchanged. Then it is easy to show that until the full insurance point is reached feasible expansions of the welfare state must increase the overall level of insurance available to individuals. First note that the expected consumption of person i is just pY and so changes in τ and r will have no effect on individuals' expected consumption. However it is straightforward to verify that whenever $Y - r > \tau$, feasible expansions of the welfare state must reduce individuals' consumption variance. Therefore despite the crowding out effect that increases in the size of the welfare state have on the intra-family income transfer, there is nevertheless a role for public insurance in reducing the consumption variance of risk averse individuals below the level that might be achieved if only family transfers were available.

In what follows we seek to identify the main features of the socially optimal public insurance scheme under the current scenario. In doing so we assume that the social planner takes the existence of intra-family insurance arrangements as given and knows that family members make income transfers according to $F = \frac{Y - r - \tau}{2}$. The planner must

therefore choose values of τ and r that maximize individual expected utility subject to maintaining a balanced budget. The socially optimal values of τ and r must therefore be the unique solution to

$$\max_{\tau, r} \left\{ (1-p)^2 u(\tau) + 2(1-p)pu \left(\frac{Y-r+\tau}{2} \right) + p^2 u(Y-r) \right\} \quad (S)$$

subject to

$$pr - (1-p)\tau = 0.$$

Note that we have substituted $F = \frac{Y-r-\tau}{2}$ into the expression for individual expected utility in order to yield the objective function for the problem S. Let λ represent the Lagrange multiplier for the government's budget constraint. Then the first order conditions for S are given by

$$\tau : (1-p)^2 u'(\tau) + (1-p)pu' \left(\frac{Y-r+\tau}{2} \right) - \lambda(1-p) = 0$$

and

$$r : -p(1-p)u' \left(\frac{Y-r+\tau}{2} \right) - p^2 u'(Y-r) + \lambda p = 0.$$

Combining these conditions and eliminating λ implies that the socially optimal levels of τ and r must satisfy

$$\frac{(1-p)^2 u'(\tau) + (1-p)pu' \left(\frac{Y-r+\tau}{2} \right)}{(1-p)pu' \left(\frac{Y-r+\tau}{2} \right) + p^2 u'(Y-r)} = \frac{1-p}{p}. \quad (4.4)$$

Condition (4.4), along with the government's budget constraint, characterize the values of τ and r that maximize social welfare under the current scenario. Condition (4.4) carries the standard economic interpretation. The left hand side represents the slope of an individual's indifference curve in (r, τ) -space, namely $\frac{dr}{d\tau}|_{EU \text{ const.}}$ evaluated at $F = \frac{Y-r-\tau}{2}$, while the right hand side represents the locus of τ and r for which the government's budget is balanced, namely $\frac{dr}{d\tau}|_{B1}$, also in (r, τ) -space. We can now

present our next result.

Proposition 2. When there are no moral hazard problems and family insurance contracts are enforceable, (i) the socially optimal public insurance scheme must choose τ and r so that $Y - r = \tau = pY$; and (ii) $\tau = r = 0$ can never be socially optimal. Therefore under the stated hypotheses there is always a role for public insurance to improve social welfare even when insurance is also provided within family networks. Furthermore the size of the welfare state matters for social welfare.

Proof. To see that the optimal public insurance scheme sets $Y - r = \tau$, re-arrange (4.4) to get

$$pu' \left(\frac{Y - r + \tau}{2} \right) - pu' (Y - r) = (1 - p) u' \left(\frac{Y - r + \tau}{2} \right) - (1 - p) u' (\tau). \quad (4.5)$$

Now suppose $Y - r > \tau$. Then $\tau < \frac{Y - r + \tau}{2} < Y - r$ and we must have

$$pu' \left(\frac{Y - r + \tau}{2} \right) - pu' (Y - r) > 0$$

and

$$(1 - p) u' \left(\frac{Y - r + \tau}{2} \right) - (1 - p) u' (\tau) < 0,$$

which contradicts condition (4.5). Likewise suppose $Y - r < \tau$. Then $Y - r < \frac{Y - r + \tau}{2} < \tau$ and we must have

$$pu' \left(\frac{Y - r + \tau}{2} \right) - pu' (Y - r) < 0$$

and

$$(1 - p) u' \left(\frac{Y - r + \tau}{2} \right) - (1 - p) u' (\tau) > 0,$$

which also contradicts (4.5). Therefore the optimal public insurance scheme must set $Y - r = \tau$. To see that $\tau = pY$, substitute $Y - r = \tau$ into the government's budget constraint and re-arrange. Finally, we have $\tau = pY \neq 0$ which also implies $r = (1 - p)Y \neq 0$. ■

Proposition 2 establishes that under the social welfare maximizing public insurance

scheme family members would consume their expected income with certainty and there would be no further benefits from making income transfers between family members. Intra-family income transfers would therefore be completely eliminated under the optimal public insurance scheme. We offer the following interpretation of this result. Because family insurance occurs within a small group of people (in our simple model the family consists of just two people), the family by itself is unable to completely smooth the consumption of family members across all states of the world. Therefore family members continue to experience some consumption variation, even after taking account of income transfers within the family. However under a public insurance scheme a risk neutral government is able to implement income transfers across a much larger group of people. For instance, suppose the *total* realized income of a particular family was zero. Then under a public insurance arrangement it would be possible to implement a transfer of income from a family that received some positive total income towards the family with zero income. In the extreme case where such transfers create no incentive problems, as is the case in our simple model, a risk neutral government would be able to offer each family member a guaranteed level of consumption in each state of the world, this consumption being equal to the individuals' expected consumption in the absence of the public insurance scheme. The public insurance scheme therefore preserves each person's expected consumption but reduces their consumption variance. Since individuals are risk averse this must improve their welfare. Note that under the optimal public insurance scheme $r = (1 - p)Y$, where r also has the interpretation of being an 'insurance premium' and $(1 - p)Y$ is an individual's expected loss. Therefore under the optimal public insurance scheme individuals should receive full insurance at a fair premium.

The model developed so far, although very simple, serves to highlight the possible consequences of public insurance for family income transfers and demonstrates the role a public insurance scheme can play in improving welfare. We have shown that when there are no moral hazard problems and family members enter into binding contracts, it is in fact socially optimal for family insurance arrangements to be completely replaced

by the public insurance scheme. However our analysis ignores some of the possible incentive problems that can arise due to a public insurance scheme. In the next section we consider a more realistic scenario in which each person's probability distribution of income is contingent upon their own level of effort. We assume that the government is unable to observe individual effort but that family members are able to observe each other.

4.3 The Model with Endogenous Income Distributions and Effort Observable Within the Family

In what follows we allow the *likelihood* of an individual receiving a particular income to depend upon that individual's level of effort. This could reflect, say, the time and energy put into finding employment or how hard a person works at a given job. Also we assume that the government is unable to observe individuals' effort but that family members are able to observe each other's actions. Family members H and W continue to receive stochastic incomes drawn from the finite set $\{0, Y\}$. This time let e and \tilde{e} represent the effort levels of H and W respectively. Then the probability that H receives Y is $p(e)$, while the probability that H receives zero is $1 - p(e)$. Also, the probability that W receives Y is $p(\tilde{e})$, while the probability that W receives zero is $1 - p(\tilde{e})$.

We assume $p' > 0$ and $p'' < 0$. In other words greater effort raises the probability of being employed and receiving positive income but does so at a diminishing rate. We also assume $p'(0) = +\infty$. As we shall see later this assumption is required in order to ensure that, whenever $Y - r > \tau$, there is always an interior solution for effort (i.e. $e > 0$). Note that whenever $Y - r \leq \tau$ family members choose exactly zero effort (see lemma 1 below). Therefore an interior solution for effort, whenever $Y - r > \tau$, is essential if the problem of designing the optimal welfare state is not to become redundant. If individuals chose zero effort no matter what public insurance scheme were in place then the socially optimal public insurance scheme would simply entail providing full public insurance, i.e.

$$Y - r = \tau.$$

Taxes, transfers and the intra-family income transfer are as specified in the previous section. Finally, family members' utility function, $u(\cdot)$, remains unchanged. Throughout this section and the remainder of this chapter we simplify notation and write p for $p(e)$ and \tilde{p} for $p(\tilde{e})$. We can now write the expected utility of H and W as

$$EU^H = (1-p)(1-\tilde{p})u(\tau) + (1-p)\tilde{p}u(\tau+F) + p(1-\tilde{p})u(Y-r-F) + p\tilde{p}u(Y-r) - e$$

and

$$EU^W = (1-\tilde{p})(1-p)u(\tau) + \tilde{p}(1-p)u(Y-r-F) + (1-\tilde{p})pu(\tau+F) + \tilde{p}pu(Y-r) - \tilde{e},$$

respectively. Under this setup the government's budget constraint is given by

$$-(1-p)(1-\tilde{p})2\tau + (1-p)\tilde{p}(r-\tau) + p(1-\tilde{p})(r-\tau) + p\tilde{p}2r = 0,$$

which reduces to

$$pr - (1-p)\tau = 0$$

whenever $p = \tilde{p}$.

Under this expanded framework the family's problem becomes that of choosing e, \tilde{e} and F , taking the tax schedule, $\{\tau, r\}$, as given. Since family members are identical then symmetry implies that any solution to the family's decision problem must entail $e = \tilde{e}$. We assume that effort can never be strictly negative. Therefore given τ and r we can express the family's problem as

$$\max_{e, F} \{ (1-p)^2 u(\tau) + (1-p)pu(\tau+F) + p(1-p)u(Y-r-F) + p^2 u(Y-r) - e \},$$

which yields the first order conditions

$$e : \{-2(1-p)u(\tau) + (1-2p)[u(\tau+F) + u(Y-r-F)] + 2pu(Y-r)\}p' - 1 = 0 \quad (4.6)$$

and

$$\begin{aligned} F : & (1-p)p[u'(\tau+F) - u'(Y-r-F)] \\ & + \{-2(1-p)u(\tau) + (1-2p)[u(\tau+F) + u(Y-r-F)] + 2pu(Y-r)\}p' - 1 \} \frac{\partial e}{\partial F} \\ & = 0. \end{aligned} \quad (4.7)$$

Substituting (4.6) into (4.7) implies

$$F : (1-p)p[u'(\tau+F) - u'(Y-r-F)] = 0 \quad (4.8)$$

and hence

$$F = \frac{Y-r-\tau}{2}. \quad (4.9)$$

Condition (4.6) simply says that individuals choose effort so that their marginal expected utility benefit of effort, $MEUB(e)$, equals their marginal expected utility cost of effort, $MEUC(e) = 1$.⁴ Condition (4.9) just says that family members offer each other full insurance whenever effort is observable within the family. This is as we would expect since family members are assumed to be identical and there are no incentive problems within the family. Substituting (4.9) into (4.6) implies that any solution to (4.6) can be written as

$$e = e(\tau, r), \quad (4.10)$$

⁴Note that since $MEUB(e)$ is monotonically decreasing in effort, then ensuring the existence of an interior solution for effort, whenever $Y-r > \tau$, requires that for any $Y-r > \tau$ the marginal expected utility benefit of effort, evaluated at zero effort, must be strictly greater than one, i.e. $MEUB(0) > 1$. But note that the expression in $\{\cdot\}$ in condition (4.6) approaches zero as $Y-r > \tau \rightarrow \tau$. Therefore in order to ensure that $MEUB(0)$ does not also approach zero as $Y-r > \tau \rightarrow \tau$ we have assumed $p'(0) = +\infty$.

for some function e of the exogenous parameters τ and r . Therefore for any τ and r conditions (4.9) and (4.10) characterize the privately optimal choices of e and F , when effort is observable within the family.

We now examine the effect of *feasible* changes in the size of the welfare state on the level of the intra-family income transfer. First totally differentiate the budget constraint, $pr - (1 - p)\tau = 0$, taking account of the effects of changes in τ and r on e and hence on p . Re-arrange the resulting expression to get

$$\frac{dr}{d\tau|_{B2}} = \frac{(1 - p) - (r + \tau)p' \frac{\partial}{\partial \tau} e(\tau, r)}{p + (r + \tau)p' \frac{\partial}{\partial r} e(\tau, r)}, \quad (4.11)$$

where p and p' are evaluated at $e = e(\tau, r)$, the solution to (4.6) for some function e of τ and r . Note that

$$\frac{\partial}{\partial \tau} e(\tau, r) = \frac{2(1 - p)u'(\tau) - \frac{1}{2}(1 - 2p)[u'(\tau + F) + u'(Y - r - F)]}{\frac{p''}{(p')^2} + 2p'[u(\tau) - u(\tau + F) - u(Y - r - F) + u(Y - r)]} \quad (4.12)$$

and

$$\frac{\partial}{\partial r} e(\tau, r) = \frac{2pu'(Y - r) + \frac{1}{2}(1 - 2p)[u'(\tau + F) + u'(Y - r - F)]}{\frac{p''}{(p')^2} + 2p'[u(\tau) - u(\tau + F) - u(Y - r - F) + u(Y - r)]}. \quad (4.13)$$

Condition (4.11) simply tells us, for any unit change in τ , the change in r that would be required to keep the government's budget in balance when income distributions are endogenous and effort is observable within the family.⁵ We can now state the following result.

⁵Expression (4.11) appears similar to equation (5) of Arnott and Stiglitz (1991, p. 182). However it is important to bear in mind the key difference between their expression and ours. Expression (5) of their paper characterizes the changes in the insurance premium, β , and the insurance payout, α , for which zero profits continue to be made *in the absence of family insurance*. Hence their derivatives $\frac{\partial e}{\partial \alpha}$ and $\frac{\partial e}{\partial \beta}$ do not take account of the effects that changes in α and β have on effort that occur via changes in the family income transfer. Also $\frac{\partial e}{\partial \alpha}$ and $\frac{\partial e}{\partial \beta}$ in their case can be shown to be unambiguously negative. On the other hand, equation (4.11) of this chapter takes account of the presence of intra-family insurance arrangements and of the effect that changes in τ and r have on effort, both directly and via changes in F . Note that in our case both $\frac{\partial e}{\partial \tau}$ and $\frac{\partial e}{\partial r}$ are ambiguous in sign.

Proposition 3. When the government is unable to observe individual effort but effort is observable within the family, and when family insurance contracts are enforceable, a feasible expansion of the welfare state will crowd out (crowd in) family transfers *if and only if* $\frac{dr}{d\tau}|_{B2} > -1$ ($\frac{dr}{d\tau}|_{B2} < -1$). Also a feasible expansion in the welfare state will have no effect on family transfers *if and only if* $\frac{dr}{d\tau}|_{B2} = -1$.

Proof. Differentiate $F = \frac{Y-r-\tau}{2}$ with respect to τ , taking account of the change in r required to maintain a balanced budget. This implies

$$\frac{dF}{d\tau} = \frac{\partial F}{\partial \tau} + \frac{\partial F}{\partial r} \frac{dr}{d\tau}|_{B2} = -\frac{1}{2} - \frac{1}{2} \frac{dr}{d\tau}|_{B2}. \quad (4.14)$$

From (4.14) it follows that

$$\frac{dF}{d\tau} \begin{matrix} \leq \\ > \end{matrix} 0 \Leftrightarrow \frac{dr}{d\tau}|_{B2} \begin{matrix} \geq \\ < \end{matrix} -1.$$

■

Proposition 3 describes the effect on F of a feasible expansion in the size of the welfare state when income distributions are endogenous and effort is observable within the family. It demonstrates that unless effort incentives are such that an increase in the public transfer, τ , requires a *reduction* in the level of the tax, r , by an equal or greater amount in order to maintain a balanced budget, then any feasible increase in τ must lead to a reduction in the level of income transferred between family members.

We now examine certain features of the socially optimal welfare state when income distributions are endogenous and effort is observable within the family. In doing so it is important to take account of the fact that family members choose e according to (4.6) and make income transfers according to $F = \frac{Y-r-\tau}{2}$. We follow Arnott and Stiglitz (1991) in arguing that the social planner knows the relationship between p and e . Therefore by observing how p responds to changes in τ and r , a social planner can implicitly take account of the fact that e also responds to τ and r according to (4.10), the solution to (4.6). We also assume that the social planner knows that family members make income transfers according to $F = \frac{Y-r-\tau}{2}$. In this case, the social welfare maximizing levels of τ

and r must be the solution to

$$\max_{\tau, r} \left\{ (1-p)^2 u(\tau) + 2(1-p)pu \left(\frac{Y-r+\tau}{2} \right) + p^2 u(Y-r) - e \right\} \quad (S_{EO})$$

subject to

$$pr - (1-p)\tau = 0$$

and

$$e = e(\tau, r),$$

where $p = p(e)$. Let λ represent the Lagrange multiplier on the government's budget constraint. Then the Lagrangian for the problem S_{EO} is given by

$$\begin{aligned} L = & (1-p)^2 u(\tau) + 2(1-p)pu \left(\frac{Y-r+\tau}{2} \right) + p^2 u(Y-r) - e(\tau, r) \\ & + \lambda(pr - (1-p)\tau), \end{aligned}$$

where $p = p(e)$ and $e = e(\tau, r)$. The first order conditions for this problem are given by

$$\begin{aligned} \tau : & (1-p)^2 u'(\tau) + (1-p)pu' \left(\frac{Y-r+\tau}{2} \right) \\ & + \left\{ \left[-2(1-p)u(\tau) + 2(1-2p)u \left(\frac{Y-r+\tau}{2} \right) + 2pu(Y-r) \right] p' - 1 \right\} \frac{\partial}{\partial \tau} e(\tau, r) \\ = & \lambda \left[(1-p) - (r+\tau)p' \frac{\partial}{\partial \tau} e(\tau, r) \right] \end{aligned}$$

and

$$\begin{aligned} r : & -p(1-p)u' \left(\frac{Y-r+\tau}{2} \right) - p^2 u'(Y-r) \\ & + \left\{ \left[-2(1-p)u(\tau) + 2(1-2p)u \left(\frac{Y-r+\tau}{2} \right) + 2pu(Y-r) \right] p' - 1 \right\} \frac{\partial}{\partial r} e(\tau, r) \\ = & -\lambda \left[p + (r+\tau)p' \frac{\partial}{\partial r} e(\tau, r) \right]. \end{aligned}$$

Substituting from equation (4.6) these expressions reduce to

$$\tau : (1-p)^2 u'(\tau) + (1-p) p u' \left(\frac{Y-r+\tau}{2} \right) = \lambda \left[(1-p) - (r+\tau) p' \frac{\partial}{\partial \tau} e(\tau, r) \right]$$

and

$$r : p(1-p) u' \left(\frac{Y-r+\tau}{2} \right) + p^2 u'(Y-r) = \lambda \left[p + (r+\tau) p' \frac{\partial}{\partial r} e(\tau, r) \right].$$

Finally, combining these two conditions and eliminating λ gives us

$$\frac{(1-p)^2 u'(\tau) + (1-p) p u' \left(\frac{Y-r+\tau}{2} \right)}{p(1-p) u' \left(\frac{Y-r+\tau}{2} \right) + p^2 u'(Y-r)} = \frac{(1-p) - (r+\tau) p' \frac{\partial}{\partial \tau} e(\tau, r)}{p + (r+\tau) p' \frac{\partial}{\partial r} e(\tau, r)}. \quad (4.15)$$

Condition (4.15), along with the budget constraint, characterize the values of τ and r that maximize social welfare when effort is observable within the family. We can contrast (4.15) with (4.4) and see that (4.15) carries a similar economic interpretation to that of (4.4). The left hand side of (4.15) represents the slope of an individual's indifference curve in (r, τ) - space, i.e. $\frac{dr}{d\tau}|_{EU \text{ const.}}$ evaluated at $F = \frac{Y-r-\tau}{2}$, while the right hand side represents the locus of τ and r for which the government's budget is balanced, namely $\frac{dr}{d\tau}|_{B2}$, also in (r, τ) - space. This time, however, $\frac{dr}{d\tau}|_{B2}$ takes account of the effect of changes in τ and r on effort and on the probabilities. Contrasting (4.15) with (4.4) demonstrates that taking account of the role of effort and of the incentive effects of a public insurance scheme must affect the optimal design of such a scheme. We explore this issue in greater detail but in order to do so we will first require the following lemma.

Lemma 1. Suppose income distributions are endogenous, effort is observable within the family, and family insurance contracts are enforceable. (i) If the public insurance scheme sets $Y - r = \tau$ then family members choose $F = 0$ and $e = \tilde{e} = 0$. (ii) If the public insurance scheme sets $Y - r < \tau$, then family members also choose $e = \tilde{e} = 0$. (iii) In the first case family members consume $p(0)Y$ with certainty, while in the second case family members' expected consumption is given by $p(0)Y$.

Proof. (i) If $Y - r = \tau$ then $F = 0$ follows directly from (4.9). Suppose $Y - r = \tau = \bar{Y}$ and $F = 0$. Then $EU^H = u(\bar{Y}) - e$ and $EU^W = u(\bar{Y}) - \tilde{e}$. It follows that greater effort will confer no expected utility benefit to the individual making the effort but will confer an expected utility cost. Maximizing expected utility will therefore require setting effort as low as possible. But since effort can never be strictly negative, individuals will be constrained to choosing zero effort.

(ii) If $Y - r < \tau$ then this must make effort incentives even worse than they were when $Y - r = \tau$. When $Y - r < \tau$ the public insurance scheme rewards individuals for being unemployed and punishes them for being employed. Individuals must therefore continue to choose $e = \tilde{e} = 0$, just as they did when $Y - r = \tau$. To see this algebraically, note that when $Y - r = \tau = \bar{Y}$ and $F = 0$, conditions (4.12) and (4.13) imply $\frac{\partial e}{\partial \tau} = \frac{\partial e}{\partial r} = \frac{u'(\bar{Y})}{\frac{p''}{(p')^2}} < 0$. Therefore starting from a position where $Y - r = \tau$, individuals would always wish to reduce effort if they experienced a change in τ and/or r that lead to $Y - r < \tau$. But remember that individuals chose zero effort when $Y - r = \tau$ and $F = 0$. Since effort can never be strictly negative, individuals must therefore continue choosing zero effort if $Y - r < \tau$.

(iii) When $Y - r = \tau$ and $F = 0$ family members consume $Y - r = \tau$ in each state. But substituting $Y - r = \tau$ into the budget constraint implies $Y - r = \tau = p(e)Y$ and so family members consume $p(e)Y$ with certainty whenever $Y - r = \tau$. Also taking account of the budget constraint gives us $p(e)Y$ as the expression for family members' expected consumption. Finally, substitute $e = \tilde{e} = 0$ into $p(e)Y$ to get our result. ■

We can now state the following result.

Proposition 4. Suppose the government is unable to observe individual effort but effort is observable within the family, and family members enter into binding insurance contracts. (i) Assuming $p' > 0$, $p'' < 0$ and $p'(0) = +\infty$, then under the optimal public insurance scheme τ and r must be chosen so that $Y - r > pY > \tau$. (ii) $\tau = r = 0$ can never be socially optimal and so some government intervention is always welfare improving. Therefore under the stated hypotheses maximizing social welfare requires the

provision of some *partial* level of public insurance.

Proof. (i) Suppose $Y - r = \tau$. Then (4.15) becomes

$$\frac{(1-p)}{p} = \frac{(1-p) - Y p' \frac{\partial}{\partial \tau} e(\tau, r)}{p + Y p' \frac{\partial}{\partial \tau} e(\tau, r)}. \quad (4.16)$$

Also from the budget constraint we have $Y - r = \tau = pY$ and from lemma 1 we have $F = 0$. Then from (4.12) and (4.13) we must have $\frac{\partial}{\partial \tau} e(\tau, r) = \frac{\partial}{\partial r} e(\tau, r) = \frac{u'(pY)}{\frac{p''}{(p')^2}} < 0$. But this means that (4.16) entails a contradiction and so $Y - r = \tau$ can never be socially optimal. However $Y - r < \tau$ must be worse in terms of social welfare than $Y - r = \tau$. Suppose $Y - r < \tau$ and the budget constraint is satisfied. Then individuals' expected utility must be lower than their expected utility when $Y - r = \tau$. The proof is straightforward. First, from lemma 1, we saw that when $Y - r < \tau$ individuals will choose zero effort and individuals' expected consumption will be given by $p(0)Y$. However when $Y - r < \tau$ individuals now experience an increase in their consumption variation compared to the $Y - r = \tau$ case, regardless of the level of F . (Lemma 1 implies that consumption variation is zero when $Y - r = \tau$). Therefore when $Y - r < \tau$ individuals exert the same level of effort, receive the same expected consumption, but experience an increase in their consumption variance compared to the situation where $Y - r = \tau$. Since family members are risk averse, such a situation must reduce their expected utility. However in both cases the budget constraint was assumed to be satisfied and so it follows that setting $Y - r < \tau$ must reduce social welfare *vis-a-vis* setting $Y - r = \tau$. But since $Y - r = \tau$ cannot be optimal, then under the stated hypotheses the optimal welfare state must entail $Y - r > \tau$. Finally, to see that $Y - r > pY > \tau$, substitute $\tau = Y - r - X$, where $X > 0$, into the budget constraint and re-arrange to get $Y - r = pY + (1-p)X$ or $Y - r > pY$. Since $\tau + X = Y - r$, then $\tau + X = pY + (1-p)X$ and so $\tau = pY - pX$ or $\tau < pY$.

(ii) Suppose $\tau = r = 0$. Then condition (4.15) reduces to

$$(1 - p) \left[u'(0) - u'\left(\frac{Y}{2}\right) \right] + p \left[u'\left(\frac{Y}{2}\right) - u'(Y) \right] = 0$$

which entails a contradiction since both terms inside the square brackets must be strictly positive. Therefore the optimal welfare state cannot entail $\tau = r = 0$. ■

We interpret proposition 4 as saying that although individuals obtain insurance through their family networks, when effort is observable within the family there is *always* a clear role for the state to improve welfare through the redistributive tax system. We have established this by showing that even in the presence of family insurance arrangements and despite any possible crowding out effects caused by public insurance, zero government intervention can never be socially optimal. However proposition 4 also establishes that because of the government's moral hazard problem, any social welfare maximum must entail the state providing less than full insurance to individuals. This result is in keeping with standard results from the insurance and moral hazard literature which establish that when information is limited, insurance schemes must somehow reward good outcomes and punish bad ones. We have demonstrated that this result continues to hold even in the presence of family risk-sharing arrangements.

Interpreting our result in light of Arnott and Stiglitz (1991), those authors argued that the presence of family insurance arrangements would ultimately lead a social planner to restrict the amount of insurance provided to individuals, compared to a situation in which there was no family insurance. Here we have demonstrated that when family members insure each other and effort is observable within the family, $\tau = r = 0$ can never be socially optimal and so in this case the public insurance scheme should never be completely eliminated. Finally, in contrast with proposition 2 of Di Tella and MacCulloch (1998), our analysis suggests that the size of the welfare state indeed matters to social welfare.

4.4 The Model with Endogenous Income Distributions and Moral Hazard Within the Family

We now consider the scenario under which income distributions are endogenous and effort is unobservable within the family. The model is the same as that described in section 4.3, however since family members can no longer observe each other's effort, we follow Arnott and Stiglitz (1991) in adopting the Nash equilibrium concept. Specifically we assume that each person chooses the level of effort that maximizes his/her own expected utility, given the effort level chosen by the other person and taking the levels of τ , r and F as fixed. Also each person assumes that everyone else acts rationally and does the same thing.

Therefore under the current setup, person H chooses e to

$$\max_e \{ (1-p)(1-\tilde{p})u(\tau) + (1-p)\tilde{p}u(\tau+F) + p(1-\tilde{p})u(Y-r-F) + p\tilde{p}u(Y-r) - e \},$$

which yields the first order condition

$$e : \{ -(1-\tilde{p})u(\tau) - \tilde{p}u(\tau+F) + (1-\tilde{p})u(Y-r-F) + \tilde{p}u(Y-r) \} p' - 1 = 0. \quad (4.17)$$

Likewise, person W chooses \tilde{e} to

$$\max_{\tilde{e}} \{ (1-\tilde{p})(1-p)u(\tau) + (1-\tilde{p})pu(\tau+F) + \tilde{p}(1-p)u(Y-r-F) + \tilde{p}pu(Y-r) - \tilde{e} \},$$

which yields the first order condition

$$\tilde{e} : \{ -(1-p)u(\tau) - pu(\tau+F) + (1-p)u(Y-r-F) + pu(Y-r) \} \tilde{p}' - 1 = 0. \quad (4.18)$$

By inspection of (4.17) and (4.18) it is easy to verify that any solution to these conditions must entail $e = \tilde{e}$. In this case we write the solution to the family's problem of choosing

effort as

$$\bar{e} = \bar{e}(\tau, r, F), \quad (4.19)$$

for some function \bar{e} of the exogenous parameters τ and r and of the family's choice of F . As before equations (4.17) and (4.18) simply tell us that each family member chooses their effort level so as to equate their marginal expected utility benefit of effort with their marginal expected utility cost of effort.⁶

In choosing F family members solve the following problem, taking τ , r , e and \tilde{e} as given and taking account of conditions (4.17) and (4.18):

$$\max_F \{ (1-p)(1-\tilde{p})u(\tau) + (1-p)\tilde{p}u(\tau+F) + p(1-\tilde{p})u(Y-r-F) + p\tilde{p}u(Y-r) - e \}.$$

The first order condition for this problem is given by

$$\begin{aligned} F : & (1-p)\tilde{p}u'(\tau+F) - p(1-\tilde{p})u'(Y-r-F) \\ & + \{[-(1-\tilde{p})u(\tau) - \tilde{p}u(\tau+F) + (1-\tilde{p})u(Y-r-F) + \tilde{p}u(Y-r)]p' - 1\} \frac{\partial e}{\partial F} \\ & + \{-(1-p)u(\tau) + (1-p)u(\tau+F) - pu(Y-r-F) + pu(Y-r)\} \tilde{p}' \frac{\partial \tilde{e}}{\partial F} \\ & = 0. \end{aligned}$$

Substituting from (4.17) and (4.18) implies

$$\begin{aligned} F : & (1-p)\tilde{p}u'(\tau+F) - p(1-\tilde{p})u'(Y-r-F) \\ & + \{1 + [u(\tau+F) - u(Y-r-F)]\tilde{p}'\} \frac{\partial \tilde{e}}{\partial F} \\ & = 0, \end{aligned} \quad (4.20)$$

while taking account of the fact that $e = \tilde{e}$ in any solution to the family's problem of

⁶Following earlier reasoning the assumption of $p'(0) = +\infty$ is sufficient to ensure we have interior solutions for the effort levels.

choosing effort implies

$$\begin{aligned}
F & : (1-p)p[u'(\tau+F) - u'(Y-r-F)] \\
& \quad + \{1 + [u(\tau+F) - u(Y-r-F)]p'\} \frac{\partial e}{\partial F} \\
& = 0,
\end{aligned} \tag{4.21}$$

where

$$\frac{\partial e}{\partial F} = \frac{pu'(\tau+F) + (1-p)u'(Y-r-F)}{\frac{p''}{(p')^2} + p'[u(\tau) - u(\tau+F) - u(Y-r-F) + u(Y-r)]} < 0. \tag{4.22}$$

Condition (4.21) implies that when there is moral hazard within the family, the optimal intra-family income transfer depends not only on the levels of τ and r set by the government but also on family members' choice of effort. This contrasts with our two previous scenarios, the cases where effort was observable within the family and where income distributions were exogenous. Under both those scenarios we obtained $F = \frac{Y-r-\tau}{2}$ so that family members offered each other full insurance and the level of F was independent of the level of effort. Therefore when there is moral hazard within the family any feasible expansion of the welfare state must have two effects within the household. First, any changes in τ and r will affect family members' net incomes in each state of the world, thereby affecting each person's variation in net incomes across states of the world. It also affects the difference between family members' net incomes in any given state of the world. This must have an effect on the intra-family income transfer by affecting the scope for intra-family risk-sharing. In addition however, when there is moral hazard within the family, changes to τ and r must also affect the family's optimal choice of effort and this in turn will exert an additional effect on the intra-family income transfer. Therefore the overall impact of a feasible expansion of the welfare state on the intra-family income transfer must depend on the combination of both these effects.

From condition (4.21) we can write the solution to the family's problem of choosing

F as

$$\bar{F} = \bar{F}(\tau, r, e), \quad (4.23)$$

for some function \bar{F} of the parameters τ and r and of the family's choice of e . Note that by combining (4.19) and (4.23) we can write the family's choices entirely in terms of the parameters of the problem. Therefore for some functions e^* and F^* , let

$$e^* = e^*(\tau, r) \quad (4.24)$$

and

$$F^* = F^*(\tau, r) \quad (4.25)$$

represent, respectively, the levels of effort and of the intra-family income transfer that will be privately optimal for a given tax schedule, $\{\tau, r\}$, when income distributions are endogenous and there is moral hazard within the family.

From (4.21) it is possible to show that we must have

$$F^* < \frac{Y - r - \tau}{2}. \quad (4.26)$$

Suppose $u'(\tau + F) - u'(Y - r - F) < 0$. Then (4.21) implies

$$1 + [u(\tau + F) - u(Y - r - F)]p' < 0,$$

since $\frac{\partial e}{\partial F} < 0$, which in turn implies $u(\tau + F) - u(Y - r - F) < -\frac{1}{p'} < 0$, a contradiction.

Suppose $u'(\tau + F) - u'(Y - r - F) = 0$. Then (4.21) implies

$$1 + [u(\tau + F) - u(Y - r - F)]p' = 0$$

which in turn implies $u(\tau + F) - u(Y - r - F) = -\frac{1}{p'} < 0$, also a contradiction. It

therefore follows that we must have

$$u'(\tau + F) - u'(Y - r - F) > 0$$

and so (4.26) follows as a result. This tells us that when there is moral hazard within the family, family members offer each other less than full insurance.

Total differentiation of the budget constraint, $p(e^*)r - (1 - p(e^*))\tau = 0$, implies

$$\frac{dr}{d\tau|_{B3}} = \frac{(1 - p) - (r + \tau)p' \frac{\partial}{\partial \tau} e^*(\tau, r)}{p + (r + \tau)p' \frac{\partial}{\partial r} e^*(\tau, r)}, \quad (4.27)$$

where p and p' are evaluated at $e^*(\tau, r)$. Condition (4.27) tells us, for any unit change in τ , the change in r that would be required to keep the government's budget in balance when income distributions are endogenous and there is moral hazard within the family. We will make use of expression (4.27) later. We now present some features of the socially optimal welfare state when there is moral hazard within the family, however first we present the following lemma.

Lemma 2. Suppose income distributions are endogenous, effort is unobservable within the family, and family insurance contracts are enforceable. (i) If the public insurance scheme sets $Y - r = \tau$, then family members choose $F^* = 0$ and $e^* = 0$. (ii) If the public insurance scheme sets $Y - r < \tau$, then family members also choose $e^* = 0$. (iii) In the first case family members consume $p(0)Y$ with certainty, while in the second case family members' expected consumption is given by $p(0)Y$.

Proof. (i) Suppose τ and r are chosen so that $Y - r = \tau = \bar{Y}$. Then we must have $F^* = 0$ since there can be no further gains to risk-sharing within the family. Now suppose $Y - r = \tau = \bar{Y}$ and $F^* = 0$. Then we have $EU^H = u(\bar{Y}) - e$ and $EU^W = u(\bar{Y}) - \tilde{e}$. Following the argument presented in lemma 1, maximizing individual expected utility requires setting $e^* = 0$.

(ii) Suppose τ and r are chosen so that $Y - r < \tau$. This must make effort incentives

even worse than they were when $Y - r = \tau$. If $Y - r < \tau$ then the public insurance scheme rewards individuals for being unemployed and punishes them for being employed. Individuals will therefore continue to choose $e^* = 0$, just as they did when $Y - r = \tau$.

(iii) The proof is the same as for lemma 1, part (iii). ■

For any size of welfare state, let $W(\tau)$ represent the corresponding level of social welfare, expressed entirely in terms of the public transfer, τ . Then

$$\begin{aligned} W(\tau) = & (1-p)^2 u(\tau) + (1-p)pu(\tau + F^*) \\ & + p(1-p)u\left(Y - \frac{1-p}{p}\tau - F^*\right) + p^2u\left(Y - \frac{1-p}{p}\tau\right) \\ & - e^*, \end{aligned} \quad (4.28)$$

where $p = p(e^*)$, $e^* = e^*\left(\tau, \frac{1-p}{p}\tau\right)$ and $F^* = F^*\left(\tau, \frac{1-p}{p}\tau\right)$. Note that $W(\tau)$ is obtained by substituting $r = \frac{1-p}{p}\tau$ into the expected utility expression and taking account of family members' privately optimal choices of effort and the family transfer, evaluated at τ and $r = \frac{1-p}{p}\tau$. We can now present the following result.

Proposition 5. Let $W(\tau)$ be as specified in (4.28). Suppose the government is unable to observe individual effort, there is moral hazard within the family, and family members enter into binding insurance contracts. (i) Assuming $p' > 0$, $p'' < 0$ and $p'(0) = +\infty$, then under the socially optimal public insurance scheme τ and r must be chosen so that $Y - r > pY > \tau$. (ii) Let $e_0^* = e^*(0, 0)$, $F_0^* = F^*(0, 0)$, $p_0 = p(e_0^*)$ and $p'_0 = p'(e_0^*)$. Also let $\frac{\partial e^*}{\partial \tau}_0 = \frac{\partial e^*}{\partial \tau}(0, 0)$, $\frac{\partial e^*}{\partial r}_0 = \frac{\partial e^*}{\partial r}(0, 0)$, $\frac{\partial F^*}{\partial \tau}_0 = \frac{\partial F^*}{\partial \tau}(0, 0)$ and $\frac{\partial F^*}{\partial r}_0 = \frac{\partial F^*}{\partial r}(0, 0)$. Finally, let $\frac{dW}{d\tau}_0 = \frac{dW}{d\tau}(0)$. Then

$$\frac{dW}{d\tau}_0 < 0$$

if and only if

$$p_0 \frac{\partial e^*}{\partial \tau}_0 + (1 - p_0) \frac{\partial e^*}{\partial r}_0$$

$$< - \frac{p_0 (1 - p_0) \left[(1 - p_0) [u' (0) - u' (Y - F_0^*)] + p_0 [u' (F_0^*) - u' (Y)] \right.}{1 + [u (F_0^*) - u (Y - F_0^*)] p'_0} \left. + [u' (F_0^*) - u' (Y - F_0^*)] \left(p_0 \frac{\partial F_0^*}{\partial \tau} + (1 - p_0) \frac{\partial F_0^*}{\partial \tau} \right) \right]. \quad (4.29)$$

In words, if $p_0 \frac{\partial e^*}{\partial \tau} + (1 - p_0) \frac{\partial e^*}{\partial \tau}$ and $p_0 \frac{\partial F_0^*}{\partial \tau} + (1 - p_0) \frac{\partial F_0^*}{\partial \tau}$ are such that (4.29) is satisfied, then a social welfare maximum exists *locally* at the point of no government intervention.

Proof. (i) The socially optimal value of τ must be the solution to $\max_{\tau} W(\tau)$. Differentiating $W(\tau)$ with respect to τ and setting the result equal to zero implies

$$\begin{aligned} \frac{dW(\tau)}{d\tau} &= (1 - p)^2 u'(\tau) + (1 - p) p u'(\tau + F^*) \left(1 + \frac{dF^*}{d\tau} \right) \\ &\quad + p(1 - p) u' \left(Y - \frac{1 - p}{p} \tau - F^* \right) (-1) \left(\frac{dr}{d\tau|_{B3}} + \frac{dF^*}{d\tau} \right) \\ &\quad + p^2 u' \left(Y - \frac{1 - p}{p} \tau \right) (-1) \frac{dr}{d\tau|_{B3}} \\ &\quad + \left\{ \left[\begin{aligned} &-2(1 - p) u(\tau) + (1 - 2p) u(\tau + F^*) \\ &+ (1 - 2p) u \left(Y - \frac{1 - p}{p} \tau - F^* \right) + 2pu \left(Y - \frac{1 - p}{p} \tau \right) \end{aligned} \right] p' - 1 \right\} \frac{de^*}{d\tau} \\ &= 0, \end{aligned} \quad (4.30)$$

where $\frac{dr}{d\tau|_{B3}}$ is as specified in (4.27),

$$\frac{dF^*}{d\tau} = \frac{\partial F^*}{\partial \tau} + \frac{\partial F^*}{\partial r} \frac{dr}{d\tau|_{B3}} \quad (4.31a)$$

and

$$\frac{de^*}{d\tau} = \frac{\partial e^*}{\partial \tau} + \frac{\partial e^*}{\partial r} \frac{dr}{d\tau|_{B3}}. \quad (4.32)$$

Substituting from (4.17) (or (4.18)) and (4.32) into (4.30) implies

$$\frac{dW(\tau)}{d\tau} = (1 - p)^2 u'(\tau) + (1 - p) p u'(\tau + F^*) \left(1 + \frac{dF^*}{d\tau} \right)$$

$$\begin{aligned}
& +p(1-p)u'\left(Y - \frac{1-p}{p}\tau - F^*\right)(-1)\left(\frac{dr}{d\tau|_{B3}} + \frac{dF^*}{d\tau}\right) \\
& +p^2u'\left(Y - \frac{1-p}{p}\tau\right)(-1)\frac{dr}{d\tau|_{B3}} \\
& +\left\{1 + \left[u(\tau + F^*) - u\left(Y - \frac{1-p}{p}\tau - F^*\right)\right]p'\right\}\left(\frac{\partial e^*}{\partial \tau} + \frac{\partial e^*}{\partial r}\frac{dr}{d\tau|_{B3}}\right) \\
& = 0,
\end{aligned} \tag{4.33}$$

and rearranging condition (4.33) gives us

$$\begin{aligned}
\frac{dW(\tau)}{d\tau} &= (1-p)^2u'(\tau) + (1-p)pu'(\tau + F^*) \\
&+ (1-p)p\left[u'(\tau + F^*) - u'\left(Y - \frac{1-p}{p}\tau - F^*\right)\right]\frac{dF^*}{d\tau} \\
&- \left[p(1-p)u'\left(Y - \frac{1-p}{p}\tau - F^*\right) + p^2u'\left(Y - \frac{1-p}{p}\tau\right)\right]\frac{dr}{d\tau|_{B3}} \\
&+ \left\{1 + \left[u(\tau + F^*) - u\left(Y - \frac{1-p}{p}\tau - F^*\right)\right]p'\right\}\left(\frac{\partial e^*}{\partial \tau} + \frac{\partial e^*}{\partial r}\frac{dr}{d\tau|_{B3}}\right) \\
&= 0
\end{aligned} \tag{4.34}$$

as the equation that characterizes the socially optimal value of τ . Now suppose τ and r are chosen so that $Y - r = \tau = \bar{Y}$. Then $F^* = 0$ and (4.34) reduces to

$$\frac{dW(\tau)}{d\tau} = (1-p)u'(\bar{Y}) - pu'(\bar{Y})\frac{dr}{d\tau|_{B3}} + \left(\frac{\partial e^*}{\partial \tau} + \frac{\partial e^*}{\partial r}\frac{dr}{d\tau|_{B3}}\right) = 0. \tag{4.35}$$

Taking account of (4.27) and re-arranging implies

$$\frac{(1-p)u'(\bar{Y}) + \frac{\partial e^*}{\partial \tau}}{pu'(\bar{Y}) - \frac{\partial e^*}{\partial r}} = \frac{(1-p) - Yp'\frac{\partial e^*}{\partial \tau}}{p + Yp'\frac{\partial e^*}{\partial r}}. \tag{4.36}$$

Finally cross-multiplying (4.36) implies

$$\begin{aligned}
& p(1-p)u'(\bar{Y}) + \frac{\partial e^*}{\partial \tau}p + Yp'\frac{\partial e^*}{\partial r}(1-p)u'(\bar{Y}) + Yp'\frac{\partial e^*}{\partial r}\frac{\partial e^*}{\partial \tau} \\
& = (1-p)pu'(\bar{Y}) - \frac{\partial e^*}{\partial r}(1-p) - Yp'\frac{\partial e^*}{\partial \tau}pu'(\bar{Y}) + Yp'\frac{\partial e^*}{\partial \tau}\frac{\partial e^*}{\partial r},
\end{aligned}$$

or equivalently

$$p \frac{\partial e^*}{\partial \tau} + (1-p) \frac{\partial e^*}{\partial r} = -Y p' u'(\bar{Y}) \left[p \frac{\partial e^*}{\partial \tau} + (1-p) \frac{\partial e^*}{\partial r} \right],$$

or

$$Y p' u'(\bar{Y}) = -1,$$

a contradiction since we must have $Y p' u'(\bar{Y}) > 0$. Therefore $Y - r = \tau$ cannot be socially optimal. However following the same argument as in Proposition 4(i), setting $Y - r < \tau$ must be worse in welfare terms than setting $Y - r = \tau$. From lemma 2 we saw that when $Y - r < \tau$, individuals chose the same level of effort (i.e. zero effort), received the same expected consumption (i.e. $p(0)Y$) but experienced a higher consumption variation compared to the $Y - r = \tau$ case. Therefore $Y - r < \tau$ cannot be socially optimal and so under our stated hypotheses the optimal welfare state must entail $Y - r > \tau$. Since $Y - r > \tau$, following the same argument as in proposition 4(i) implies that we must have $Y - r > pY > \tau$.

(ii) From (4.34) and (4.31a) we have

$$\begin{aligned} \frac{dW(\tau)}{d\tau} &= (1-p)^2 u'(\tau) + (1-p) p u'(\tau + F^*) \\ &\quad + (1-p) p \left[u'(\tau + F^*) - u' \left(Y - \frac{1-p}{p} \tau - F^* \right) \right] \left(\frac{\partial F^*}{\partial \tau} + \frac{\partial F^*}{\partial r} \frac{dr}{d\tau|_{B3}} \right) \\ &\quad - \left[p(1-p) u' \left(Y - \frac{1-p}{p} \tau - F^* \right) + p^2 u' \left(Y - \frac{1-p}{p} \tau \right) \right] \frac{dr}{d\tau|_{B3}} \\ &\quad + \left\{ 1 + \left[u(\tau + F^*) - u \left(Y - \frac{1-p}{p} \tau - F^* \right) \right] p' \right\} \left(\frac{\partial e^*}{\partial \tau} + \frac{\partial e^*}{\partial r} \frac{dr}{d\tau|_{B3}} \right). \end{aligned} \tag{4.37}$$

Evaluating this expression at $\tau = r = 0$ (and noting that $\frac{dr}{d\tau}|_{B3}$, evaluated at $\tau = r = 0$, gives us $\frac{1-p_0}{p_0}$) implies

$$\frac{dW}{d\tau}_0 = (1-p_0) \{ (1-p_0) [u'(0) - u'(Y - F_0^*)] + p_0 [u'(F_0^*) - u'(Y)] \}$$

$$\begin{aligned}
& + (1 - p_0) p_0 [u'(F_0^*) - u'(Y - F_0^*)] \left(\frac{\partial F_0^*}{\partial \tau_0} + \frac{\partial F_0^*}{\partial r_0} \frac{1 - p_0}{p_0} \right) \\
& + \{1 + [u(F_0^*) - u(Y - F_0^*)] p'_0\} \left(\frac{\partial e^*}{\partial \tau_0} + \frac{\partial e^*}{\partial r_0} \frac{1 - p_0}{p_0} \right).
\end{aligned} \tag{4.38}$$

Finally, rearranging (4.38) implies

$$\frac{dW}{d\tau_0} \begin{matrix} \leq \\ > \end{matrix} 0$$

if and only if

$$\begin{aligned}
& p_0 \frac{\partial e^*}{\partial \tau_0} + (1 - p_0) \frac{\partial e^*}{\partial r_0} \\
\leq & - \frac{p_0 (1 - p_0) \left[(1 - p_0) [u'(0) - u'(Y - F_0^*)] + p_0 [u'(F_0^*) - u'(Y)] \right.}{1 + [u(F_0^*) - u(Y - F_0^*)] p'_0} \left. + [u'(F_0^*) - u'(Y - F_0^*)] \left(p_0 \frac{\partial F_0^*}{\partial \tau_0} + (1 - p_0) \frac{\partial F_0^*}{\partial r_0} \right) \right]}{1 + [u(F_0^*) - u(Y - F_0^*)] p'_0}.
\end{aligned} \tag{4.39}$$

■

From proposition 5(i) we see that when there is moral hazard within the family, under the socially optimal public insurance scheme τ and r must be chosen so that $Y - r > \tau$ and so individuals are offered less than full insurance. This is to be expected since we assumed that the government was unable to observe individual actions. Under the optimal public insurance scheme therefore individuals consume more when they are employed than when they are unemployed. One interpretation of proposition 5(ii) is that the presence of moral hazard within the family acts to diminish somewhat the scope for any welfare improvements via a public insurance scheme. Proposition 5(ii) tells us that if $p_0 \frac{\partial e^*}{\partial \tau_0} + (1 - p_0) \frac{\partial e^*}{\partial r_0}$ and $p_0 \frac{\partial F_0^*}{\partial \tau_0} + (1 - p_0) \frac{\partial F_0^*}{\partial r_0}$ are such that (4.29) is satisfied, then increasing the size of the welfare state, from a position of no government intervention, must be welfare reducing. Therefore under (4.29) a social welfare maximum exists *locally* at the point of no government intervention. In this case, for sufficiently

small levels of government intervention there is no scope for improving welfare via the redistributive tax system. In other words if (4.29) is satisfied then there is a social welfare (local) maximum that entails the provision of no public insurance. An alternative interpretation of proposition 5(ii) is that under (4.29), for sufficiently small levels of government intervention it is preferable for the government to offer no public insurance at all. The results of proposition 5 contrast with those of proposition 4 which established that when effort was observable within the family, $\tau = r = 0$ could never be socially optimal. In this case there was always a role for government intervention in the provision of some partial level of public insurance. On the other hand the analysis of this section suggests that when there is moral hazard within the family, the existence of a role for government in the provision social insurance will depend crucially on the effect of such insurance on effort incentives and on the intra-family income transfer.

4.5 Discussion and Areas for Further Work

The analysis of this chapter suggests a number of further lines of inquiry. A natural extension of our model is to relax the assumption that family contracts are enforceable and to consider only those risk-sharing arrangements that could be achieved through self-enforcing agreements. A proper treatment of self-enforcing family arrangements would require family members to interact repeatedly and to face income risk each period⁷. At each date, once incomes are realized and given the tax schedule, family members choose the intra-family income transfer that maximizes their discounted stream of future utilities subject to the appropriately specified self-enforcing constraints. Since family contracts are non-binding and family members can choose to renege at any time, the current family transfer can only be decided once the current state is known. At this stage each family member chooses whether to renege or to cooperate with the risk-sharing arrangement. Family members will cooperate as long as the long-term benefits of doing so outweigh any

⁷Thomas and Worrall (1984) adopt this approach in analyzing wage contracts, while Ligon, Thomas and Worrall (1997) apply this approach to analyse informal insurance arrangements.

short-term gain from renegeing. Under this extended setting we could once again explore the consequences for intra-family income transfers of expansions of the welfare state. We could also examine the optimal design of the welfare state in the presence of self-enforcing family insurance contracts. Note that we could also introduce moral hazard into this setting to see how this would affect our results. It would be interesting to compare the results of such an analysis with those of Di Tella and MacCulloch (1998), Krueger and Perri (1999) and Attanasio and Ríos-Rull (1999a, 1999b) who also examine the interaction between family and state-provided insurance schemes when family contracts are not enforceable.

We also feel that there are further lines of inquiry suggested by the work of Arnott and Stiglitz (1991). As we discussed in section 4.1, Arnott and Stiglitz establish that obtaining insurance through family networks when insurance is available through a competitive insurance market can be harmful to social welfare. As they point out, when there is moral hazard the disincentive effects of family insurance can be sufficiently strong as to lead to a reduction in the amount of insurance provided by the market and a consequent increase in the amount of insurance provided within families. This must be socially harmful because the family has fewer advantages in providing insurance than does a risk neutral firm operating in a perfectly competitive market. However these results lead us to ask whether there may also be some circumstances under which family insurance is socially beneficial. As Arnott and Stiglitz rightly point out, "...If market insurance against a given accident does not in fact exist, voluntary nonmarket insurance is unambiguously beneficial. When transactions costs are present, nonmarket insurance may be beneficial if it is provided at lower transaction cost than market insurance." (p. 186). And again, "...Market insurance is generally unavailable for the multitudinous small (but cumulatively substantial) risks faced in everyday life. How nonmarket institutions handle such risks is an important and interesting question." (p. 186).

One reason why insurance may sometimes be unavailable is adverse selection. Under adverse selection there are two types of individuals in the population, high risk types and

low risk types, and insurers are unable to observe any particular individual's type. Suppose p represents the proportion of low risks in the population. A standard result of the adverse selection literature is that there exists a critical proportion, p^* , of low risks in the population such that if $p \in (p^*, 1)$, then no market equilibrium exists, and if $p \in (0, p^*)$, the market equilibrium is a pair of separating contracts under which high risks receive full insurance at the fair premium, while low risks receive less than full insurance at the fair premium. One possible avenue for future research is to investigate whether family insurance can be beneficial under such circumstances. When p is large, the market fails to provide any insurance whatsoever and in this case family insurance may be unambiguously beneficial. However when p is small, low risks receive less than full insurance yet would prefer to be fully insured at the fair premium. Low risks may therefore find it privately beneficial to contract with family members, however the social welfare consequences of doing so will depend upon how such contracting affects the overall demand for market insurance and the terms at which such insurance is ultimately provided. Since adverse selection lies at the root of many problems of incomplete insurance markets (for instance in the case of health insurance), we feel that an analysis along such lines would be a fruitful and interesting endeavour.

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